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„Műszaki és gazdasági szakok alapozó matematikai ismereteinek e–learning alapú tananyag- és módszertani fejlesztése”

Analysis

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Contents

Introduction

This textbook contains the knowledge expected to be acquired by the students at the "College Functions" class of the Foundation Year and Intensive Foundation Semester at the University of Debrecen.

We have tried to compose the textbook such that it is a self-contained material, including many examples, figures and graphs of functions which help the students in better understanding the topic, however, this book is primarily meant to be used as a Lecture Notes connected to the "College Functions" class. Thus we suggest the students to attend the classes, and use this book as a supplementary mean of study.

The sections marked by * are sections which contain some non-compulsory subject, which are meant only for the interested readers, but will not be required for the exam at the and of semester.

In Chapter 1 we define the basic concepts connected to functions, and the most important properties of functions which we shall analyze later in the case of many types of functions.

In Chapter 2 we present some basic types of functions, like linear functions, quadratic functions, polynomial functions, power functions, exponential and logarithmic functions, along with their most important properties.

Chapter 3 is devoted to a detailed description of trigonometric functions.

The topic of Chapter 4 is the transformation of functions.

The last Chapter contains the results of the unsolved exercises of the book, which will help the students to check if their solutions lead to the right answer or not.

Debrecen, 2014. Attila Bérczes and Attila Gilányi

Chapter 1

The concept and basic properties of functions

1.1 Relations and functions

In this section we provide a precise definition of binary relations and functions, which is the mathematically correct way to introduce these concepts. However, this section is recommended mostly for the interested reader, and we give a less precise but simpler introduction of functions in the next section.

Definition 1.1. Let A be a non-empty set and $a, b \in A$ elements of A. The ordered pair (a, b) is defined by

$$(a, b) := \{a, \{a, b\}\}$$

Theorem 1.2. *Let (a, b) and (c, d) be two ordered pairs. We have*

$$(a, b) = (c, d) \iff a = c \quad \textit{and} \quad b = d$$

Definition 1.3. Let A, B be two sets. The Cartesian product $A \times B$ of A and B is the set of all ordered pairs with the first element being from A and the second element from B, i.e.

$$A \times B := \{(a, b) \,|\, a \in A,\ b \in B\} .$$

Definition 1.4. Let A, B be two non-empty sets. A **binary relation** ρ between A and be B is a subset of the Cartesian product $A \times B$, i.e. $\rho \subset A \times B$. If $(a, b) \in \rho$ then we also say that

- a **is in relation** ρ **with** b, or
- a **is assigned** b, or
- bf ρ assigns b to a.

If $A = B$ then we say that ρ is a binary relation on A.

Notation. For $(a, b) \in \rho$ we also use the notation $a\rho b$.

Example. Here we give some examples of binary relations from our earlier mathematical experience:

- The relation $\leq$ on the set of real numbers
- The realtion $\leq$ on the integers
- The congruence of triangles
- The parallelity of lines in the plane

and we also give an abstract example to illustrate the definition above:

- Let $A := \{a, b, c\}$ and $\rho \subset A \times A$, $\rho := \{(a, b), (a, c), (b, c)\}$.

Definition 1.5. Let $\rho \subset A \times B$ be a binary relation. Then

- The **domain** of ρ is the set of all elements of A which are in relation ρ with at least one element of B, i.e., the domain of ρ is defined by
$$D_\rho := \{a \in A \,|\, \exists b \in B \text{ with } (a, b) \in \rho\}$$
- The range of ρ is the set of all such elements of B which are assigned by ρ to at least one element of A, i.e. the range of ρ is defined by
$$R_\rho := \{b \in B \,|\, \exists a \in A \text{ with } (a, b) \in \rho\}$$

Definition 1.6. The binary relation $f \subset A \times B$ is called a **function** if

$$(a, b) \in f \text{ and } (a, c) \in f \implies b = c.$$

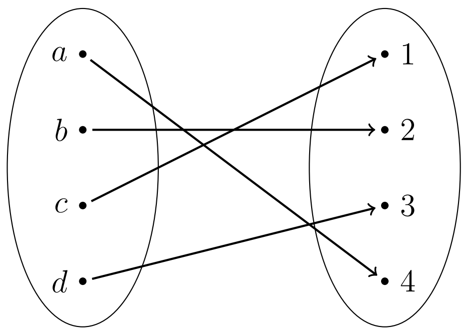
The domain and the range of the function f are the sets which are defined as the domain and the range of f considered as a relation.

Notation. If a relation f is a function, then

- instead of $f \subset A \times B$ we write $f : A \to B$ and we say that f is a function from A to B;
- instead of writing $(a, b) \in f$ or afb we shall write $f(a) = b$ and we say that f maps the element a to the element b, or b is the image of a by the function f.

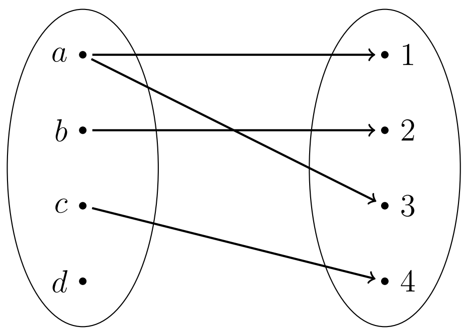
Remark. The above definition means that a function maps each element of its domain to a well-defined single element of the range.

Example. First we give an example for a function $f : A \to B$, given by a diagram, where $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$: This in fact means that $f(a) = 4$, $f(b) = 2$, $f(c) = 1$



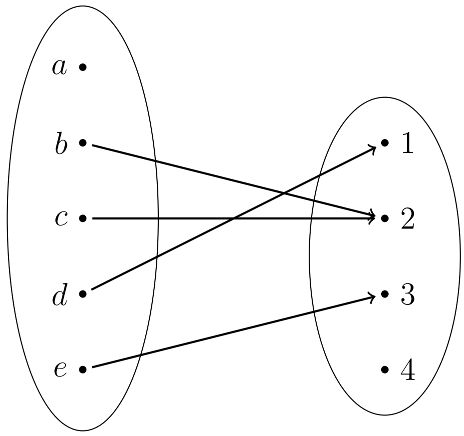
and $f(d) = 3$. The domain of f is $D_f = \{a, b, c, d\}$ and the range of f is $R_f = \{1, 2, 3, 4\}$.

Example. The following diagram defines a relation R from A to B, where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$. However this relation is not a function.



We have $aR1$, $bR2$, $cR4$, $aR3$. The domain of R is $D_f = \{a, b, c\}$ and the range of f is $R_f = \{1, 2, 3, 4\}$. We emphasize that the relation defined by the diagram is not a function, since for the same $a \in A$ we have both $aR1$ and $aR3$, and this is not allowed in the case of a function.

Example. The following diagram also defines a function $f : A \to B$ from A to B, where $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$.



We have $f(b) = 2$, $f(c) = 2$, $f(d) = 1$, $f(e) = 3$. The domain of f is $D_f = \{b, c, d, e\}$ and the range of f is $R_f = \{1, 2, 3\}$. We emphasize that the relation defined by the diagram is a function, although $f(b) = f(c) = 2$, so two different elements may have the same image by a function.

Remark. We wish to emphasize the difference between the two last examples. In the last example f is a function, even if 2 is the image of two different elements of A. On the other hand in the preceding example R is not a function since R assigns two different elements to a.

1.2 A 'friendly' introduction of functions

Definition 1.7. A relation is a correspondence between the elements of two sets.

Definition 1.8. A function is a correspondence between the elements of two sets, where to elements of the first set there is assigned or associated only one element of the second set. We use the following notations:

- For a function f from the set A to the set B we use the notation $f : A \to B$
- To say that f maps an element a of A to the element b of B we write $f(a) = b$ or $f : a \mapsto b$.

Remark. A function is in fact a relation, where to elements of the first set there is assigned or associated only one element of the second set.

Example. Pelda diagrammal megadott fuggvenyre es "nem fuggvenyre".

Definition 1.9. Let $f : A \to B$ be a function.

- The domain of f is defined by
$$D_f := \{a \in A \,|\, \exists b \in B \text{ with } f(a) = b\}$$
- The range of f is defined by
$$R_f := \{b \in B \,|\, \exists a \in A \text{ with } f(a) = b\}$$

Remark. A function f is given only if along the correspondence, the domain of the function is also given. Indeed, the functions

$$f : \mathbb{Z} \to \mathbb{Z} \qquad f(x) := x^2$$

and

$$g : \mathbb{R} \to \mathbb{R} \qquad g(x) := x^2$$

are clearly different, although the formula describing the correspondence is the same. Indeed, while $g(\sqrt{2}) = 2$, f does not assign any value to $\sqrt{2}$, and instead of $\sqrt{2}$ we could choose any element of $\mathbb{R} \setminus \mathbb{Z}$.

Convention. Whenever a function f is given by a formula, without specifying the domain of the function we agree that $f : \mathbb{R} \mapsto \mathbb{R}$ and the domain of the function is the largest subset of $\mathbb{R}$ on which the formula has sense.

Example. Let $f(x) := x^2 - 3x + 2$ be a function.

1). Compute $f(-3), f(0), f(1), f(\sqrt{2}), f(\pi), f(t+1), f(x+2), f(x^2-1)$.

2). Determine the domain of f.

3). Decide if $y_1 = 2$ and $y_2 = -3$ is in the range of f or not.

4). Determine the range of f.

Solution.

1). To compute the value of the function f given by the formula $f(x)$ taken at a number or expression a we substitute every occurrence of x in the right hand side of the formula defining f by the number or expression a.

$$f(-3) = (-3)^2 - 3 \cdot (-3) + 2 = 9 + 9 + 2 = 20$$
$$f(0) = 0^2 - 3 \cdot 0 + 2 = 0 + 0 + 2 = 2$$
$$f(1) = 1^2 - 3 \cdot 1 + 2 = 1 - 3 + 2 = 0$$
$$f(\sqrt{2}) = \sqrt{2}^2 - 3 \cdot \sqrt{2} + 2 = 2 - 3\sqrt{2} + 2 = 4 - 3\sqrt{2}$$
$$f(\pi) = \pi^2 - 3\pi + 2$$
$$f(t+1) = (t+1)^2 - 3 \cdot (t+1) + 2 = t^2 + 2t + 1 - 3t - 3 + 2 = t^2 - t$$
$$f(x+2) = (x+2)^2 - 3 \cdot (x+2) + 2 = x^2 + 4x + 4 - 3x - 6 + 2 = x^2 + x$$
$$f(x^2-1) = (x^2-1)^2 - 3(x^2-1) + 2 = x^4 - 2x^2 + 1 - 3x^2 + 3 + 2 = x^4 - 5x^2 + 6$$

2). To determine the (maximal) domain of the function f given by a formula we have to determine the maximal subset of $\mathbb{R}$ on which the formula defining f has sense.
Since $x^2 - 3x + 2$ has sense for every real number x the (maximal) domain of f is the whole set of real numbers, i.e. $D_f = \mathbb{R}$.

3). To decide if a concrete number y belongs to range of a function f given by a formula we need to solve the equation $f(x) = y$ in the variable x, where y is a given number. If the equation has a solution in the domain D_f of f then y belongs to the range of f, otherwise y does not belong to the range of f.

To decide if $y_1 = 2$ is in the range of f or not we have to check if the equation

$$f(x) = y_1$$

has a solution in D_f or not. So we solve the equation

$$x^2 - 3x + 2 = 2 \quad x^2 - 3x = 0 x_1 = 0, \quad x_2 = 3$$

Since

$$f(0) = 2$$

we see that $y_1 = 2 \in R_f$.

To decide if $y_2 = -3$ is in the range of f or not we have to check if the equation

$$f(x) = y_2$$

has a solution in D_f or not. So we solve the equation

$$x^2 - 3x + 2 = -3 \quad x^2 - 3x + 5 = 0$$

Since the discriminant $\Delta = (-3)^2 - 4 \cdot 1 \cdot 5 = -11 < 0$ the above equation has no solution and we see that $y_2 = -3 \notin R_f$.

4). To determine the range of a function f given by a formula we need to solve the equation $f(x_0) = y_0$ in the variable x_0, where y_0 is a parameter. The range of R_f of f consists of those values y_0, for which the equation $f(x_0) = y_0$ has a solution in the domain D_f of f.

Thus we shall check for which values of the parameter y_0 the equation

$$x^2 - 3x + 2 = y_0$$

has a solution in $D_f = \mathbb{R}$. The equation takes the form

$$x^2 - 3x + (2 - y_0) = 0,$$

which has a solution if and only if its discriminant is non-negative, i.e.

$$9 - 4(2 - y_0) \geq 0.$$

This is equivalent to $y_0 \geq \frac{1}{4}$, so the range of f is $R_f = \left[\frac{1}{4}, \infty\right[$.

Exercise 1.1. Let $f(x)$ be one of the functions below.

1). Compute $f(-3), f(0), f(1), f(\sqrt{2}), f(\pi), f(t+1), f(x+2), f(x^2-1)$.

2). Determine the domain of f.

3). Decide if y_1 and y_2 (given below next to the function f) is in the range of f or not.

4). Determine the range of f.

a) $f(x) = x^2 - 5x + 6, \quad y_1 = 2, \ y_2 = -1$

b) $f(x) = -x^2 + 4, \quad y_1 = 8, \ y_2 = 0$

c) $f(x) = \sqrt{x-4}, \quad y_1 = 4, \ y_2 = -2$

d) $f(x) = \sqrt{6-x}, \quad y_1 = 2, \ y_2 = -3$

e) $f(x) = 3\sqrt{x+2}, \quad y_1 = -3, \ y_2 = 3$

f) $\dfrac{1}{x-2}, \quad y_1 = -1, \ y_2 = 0$

g) $f(x) = \dfrac{x+1}{x-3}, \quad y_1 = -3, \ y_2 = 1$

h) $\dfrac{2x-3}{x+2}, \quad y_1 = 2, \ y_2 = -5$

1.3 The graph of a function

1.3.1 Cartesian coordinate system

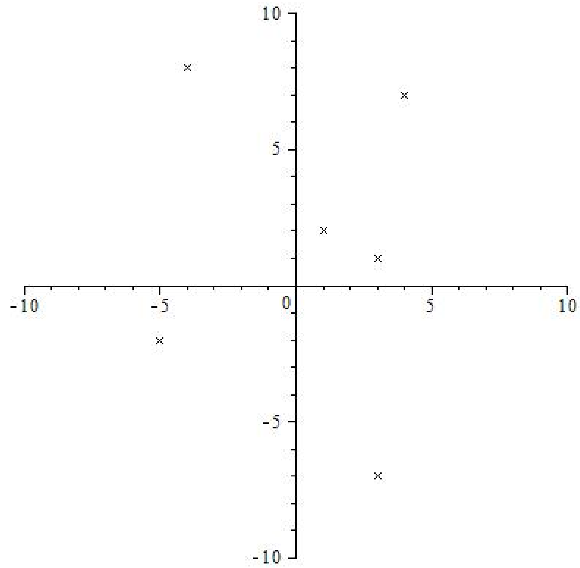
Definition 1.10. Consider two perpendicular lines on the plane, one of them being horizontal, the other one vertical. The intersection point of the two lines is called the origin, the horizontal line is called the X-axis and the vertical line is called the Y-axis. On each of these lines we represent the set of real numbers in the usual way, 0 being represented on both lines by the origin, and with the convention that on the horizontal line the positive direction is to the left, and on the vertical line the positive direction is upwards. The plane together with the two axis is called the Cartesian coordinate system.

Then any point of the plane corresponds to a pair of real numbers $(x, y) \in \mathbb{R}^2$ in the following way: x is the real number corresponding to the projection of the point to the X-axis, and y is the real number corresponding to the projection of the point to the Y-axis. Conversely, to represent a pair (x, y) as a point of the plane we draw a parallel line to the Y-axis through the point corresponding to x on the X-axis, and a parallel line to the X-axis through the point corresponding to y on the Y-axis, and the point corresponding to (x, y) will be the intersection point of these two lines.

Example. Represent the following pairs of real numbers in the Cartesian coordinate system:

$$(1, 2), (4, 7), (3, 1), (3, -7), (-5, -2), (-4, 8).$$

Solution.



1.3.2 The graph of a function

Definition 1.11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The set

$$\{(x, y) \in \mathbb{R}^2 \,|\, x \in D_f,\, y = f(x)\}$$

is called **the graph** of the function f.

The graph of a function can be represented in the so-called Cartesian coordinate system.

Graphing a function:

To draw the graph of a function $f : \mathbb{R} \to \mathbb{R}$ we proceed as follows:

1). First we compute pairs $(x, f(x))$ for several values of x.
(We also may summarize this in a table of values.)

2). Using a Cartesian coordinate system we plot the computed points $(x, f(x))$ in the plane.

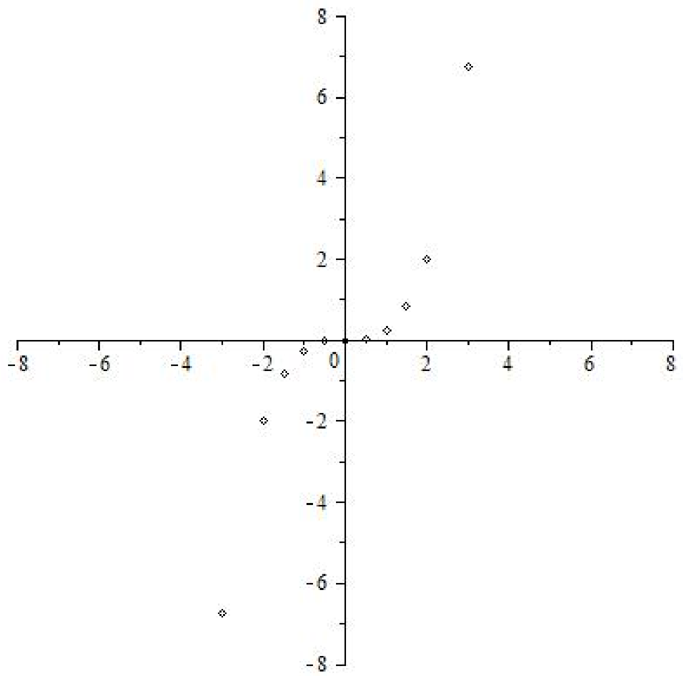
3). We connect these points by a smooth curve.

Example. Draw the graph of the function $f(x) = \frac{x^3}{4}$.

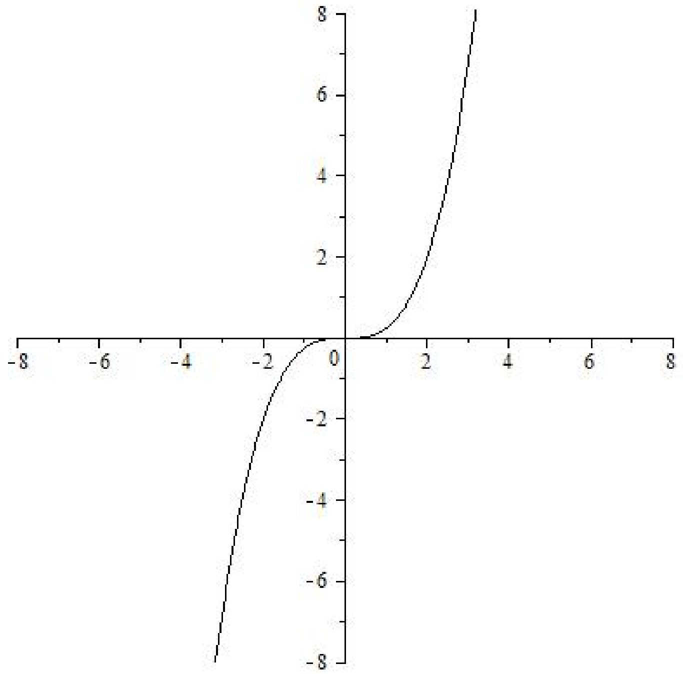
1). We compute the pairs $(x, f(x))$ for $x = -3, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 3$:

x	-3	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	3
f(x)	-27	-8	-3.375	-1	-0.125	0	0.125	1	3.375	8	27

2). We plot the points $(-3, -6.75), (-2, -2), (-1.5, -0.84375), (-1, -1), (-0.5, -0.03125),$ $(0, 0),$ $(0.5, 0.03125),$ $(1, 1), (1.5, 0.84375), (2, 2), (3, 6.75)$



3). We connect the points by a smooth curve



Exercise 1.2. Sketch the graph of the following functions:

a) $f(x) = x + 1$	b) $f(x) = 3x - 2$	c) $f(x) = -2x + 2$
d) $f(x) = x^2$	e) $f(x) = x^2 - 4$	f) $f(x) = 2x^2 - 2$
g) $f(x) = -x^2$	h) $f(x) = -\frac{1}{2}x^2$	i) $f(x) = -x^2 + 9$
j) $f(x) = \sqrt{x}$	k) $f(x) = \sqrt{x+2}$	l) $f(x) = \sqrt{x+3} - 2$
m) $f(x) = \dfrac{1}{x}$	n) $f(x) = \dfrac{3}{x}$	o) $f(x) = \dfrac{-2}{x}$
p) $f(x) = \dfrac{x+3}{x+1}$	q) $2 - \dfrac{3}{x-1}$	r) $f(x) = \dfrac{x^2+1}{3x+3}$

1.3.3 The graph of equations or relations

Similarly to the graph of functions one may represent relations or solutions of an equation in two unknowns in the Cartesian coordinate system.

Example. Represent the set of solutions of the equation

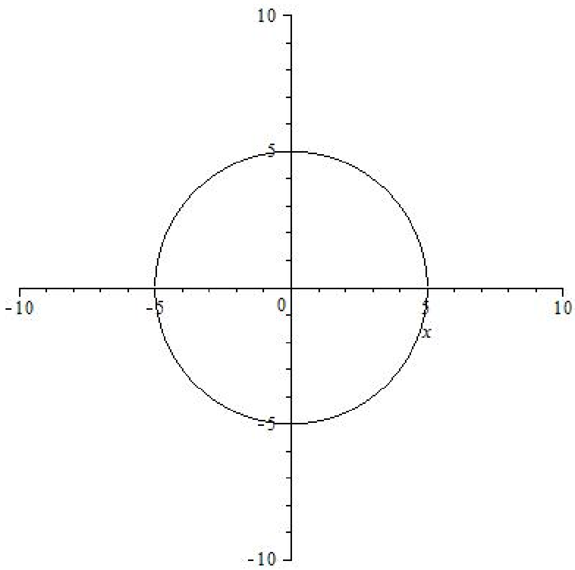
$$x^2 + y^2 = 25$$

in the Cartesian coordinate system.

Solution. We have to draw the curve corresponding to the set

$$\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 25\} \subset \mathbb{R}^2$$

in the Cartesian system, so we get

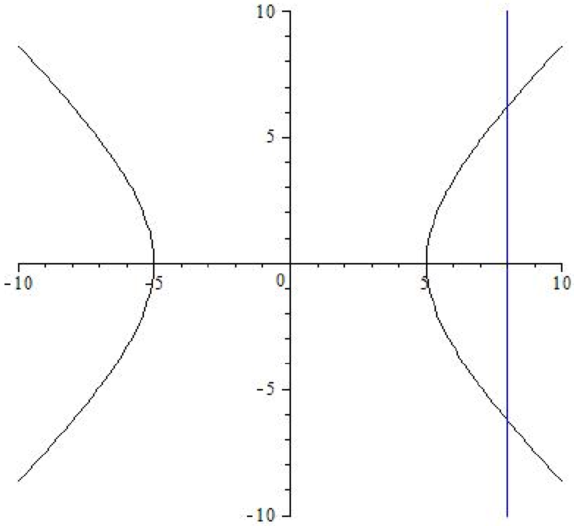


Remark. Clearly, the graph of the function $f(x)$ is the same as the graph of the equation $y = f(x)$. Thus, when there is no danger of confusion we shall not make any distinction among them.

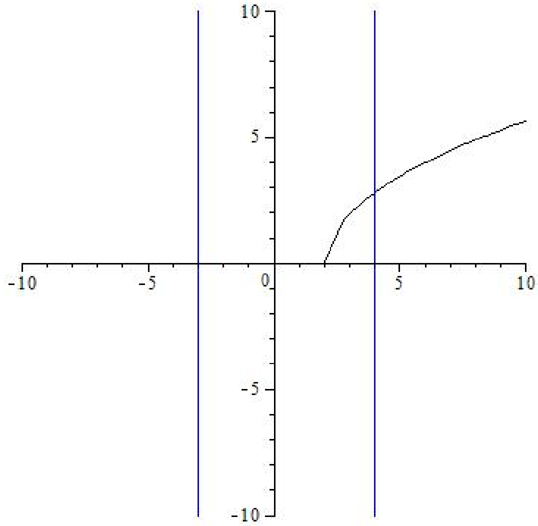
1.3.4 The vertical line test

The graph of a function may intersect or touch any vertical line in at most one point. More precisely, a curve or any set of points in the plane, drawn in the Cartesian coordinate system is the graph of a function if and only if this curve or set has at most one common point with any vertical line.

Example. The following two figures show the essential difference between the graph of a function and a curve that is not the graph of a function:



There exists a vertical line such that it intersects the curve in two points, so this is not the graph of a function

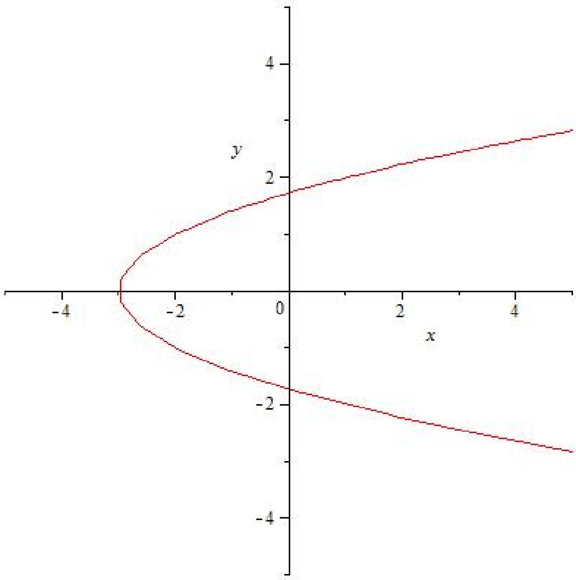


Every vertical line intersects the curve in at most one point, thus there esists a functin such that the curve is the graph of that function.

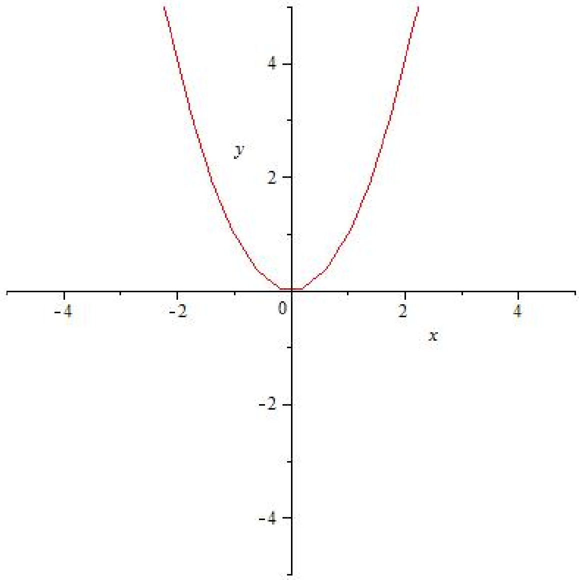
Remark. The vertical line test is a graphical method, which is very useful to understand well the concept of a function, however, in itself it is not a rigorous proving method. However, if we "see" the vertical line showing that a curve is not the graph of a function, then it is easy to write down the proof rigorously.

Exercise 1.3. Using the vertical line test decide if the following curves are graphs of a function or not:

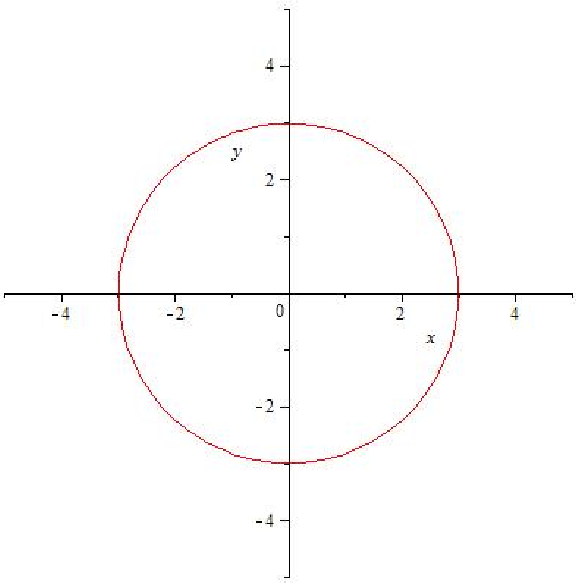
a)



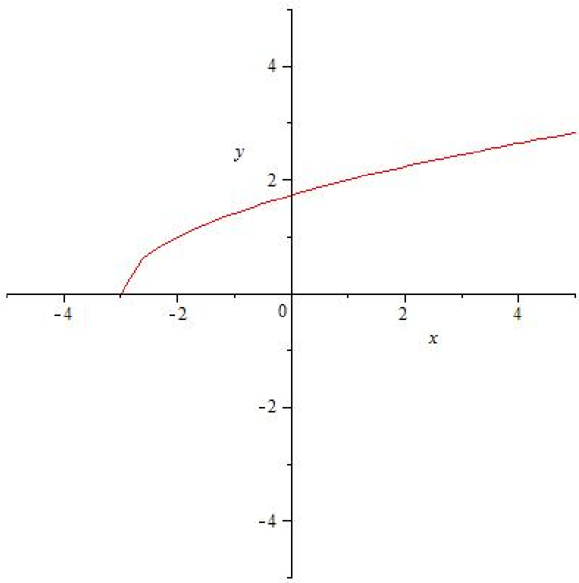
b)



c)



d)



1.3.5 The Y-intercept and the X-intercepts of functions

When drawing the graph of a function we pay special interest to the intersection of the graph with the coordinate axis.

Definition 1.12. (Intercepts of the graph of a function)

1). The intersection point of the graph of a function with the Y-axis is called the **Y-intercept** of the graph.

2). An intersection point of the graph of a function with the X-axis is called an **X-intercept** of the graph.

Remark. Note the difference on the two points of the above definition: the Y-intercept is a single point (if it exists), however there might exist more X-intercepts. Indeed, the vertical line test guarantees that the graph of a function might intersect the Y-axis (which is a vertical line) in at most one point.

Theorem 1.13. *(The Y-intercept of a function) The graph of a function f has Y-intercept if and only if it is defined in $x_0 = 0$, and it is the point $B(0, y_0)$ with $y_0 := f(0)$.*

Theorem 1.14. *(The X-intercepts of a function) The graph of a function f given by a formula $f(x)$ has X-intercepts if and only if the equation*

$$f(x) = 0 \qquad x \in D_f$$

has solutions in the domain D_f of f, and the X intercepts are the points $A_i(x_i, 0)$ for $i = 1, 2, \ldots,$ where x_i for $i = 1, 2, \ldots$ are the solutions of the above equation.

Example. Compute the Y-intercept and the X-intercepts of the graph of the function $f(x) = x^2 - 3x - 4$.

Solution. The Y intercept is the point $B(0, f(0))$, i.e. the point $B(0, -4)$ (since $f(0) = 0^2 - 3 \cdot 0 - 4$).

The X intercepts are the points having Y-coordinate zero, their X-coordinates being the solutions of the equation

$$x^2 - 4 = 0.$$

Using the "Almighty formula" these solutions are $x_1 = -1$ and $x_2 = 4$, so the X-intercepts are $A_1(-1, 0)$ and $A_2(4, 0)$.

Exercise 1.4. Compute the Y-intercept and the X-intercepts of the following functions:

a) $f(x) = 3x - 2$
b) $f(x) = -2x + 5$
c) $f(x) = 3$
d) $f(x) = x^2 - 4x + 3$
e) $f(x) = x^2 - 2$
f) $f(x) = -2x^2 + 8x - 8$
g) $f(x) = \dfrac{x-2}{x+1}$
h) $f(x) = 2 + \dfrac{3}{x-1}$
i) $f(x) = \dfrac{x^2 - 3x + 2}{x+5}$
j) $f(x) = 2^x$
k) $f(x) = 3^x$
l) $f(x) = 2^{-x}$
m) $f(x) = x^3 - 4x$
n) $f(x) = x^6 + x^2 + 2$
o) $f(x) = \dfrac{1}{x^2 - 3x + 4}$
p) $f(x) = \sqrt{x+3}$
q) $f(x) = 3 + \sqrt{x}$
r) $f(x) = -2 + \sqrt{x-3}$
s) $f(x) = |x - 2|$
t) $f(x) = |x + 1| - 3$
u) $f(x) = |x + 2| - |x - 4|$
v) $f(x) = \sin x$
w) $f(x) = \cos x$
x) $f(x) = \tan x$
y) $f(x) = \cot x$
z) $f(x) = \sin^2 x + 2$
ω) $f(x) = \cos x - \sin x$

1.4 New functions from old ones

Definition 1.15. (The sum, difference, product and quotient of functions) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by the formulas $f(x)$ and $g(x)$, and denote by D_f and D_g the domain of f and g, respectively. Put $D := D_f \cap D_g$ and $D_0 := D \setminus \{a \in \mathbb{R} \mid g(a) = 0\}$. Then

1). **the sum** $f + g$ of f and g is defined by

$$(f + g) : D \to \mathbb{R}, \quad (f + g)(x) := f(x) + g(x)$$

2). **the difference** $f - g$ of f and g is defined by

$$(f - g) : D \to \mathbb{R}, \quad (f - g)(x) := f(x) - g(x)$$

3). **the product** $f \cdot g$ of f and g is defined by

$$(f \cdot g) : D \to \mathbb{R}, \quad (f \cdot g)(x) := f(x) \cdot g(x)$$

4). **the quotient** $\frac{f}{g}$ of f and g is defined by

$$\frac{f}{g} : D_0 \to \mathbb{R}, \quad \left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)}$$

Remark. The above definition means that the sum, difference, product and quotient of two functions given by a formula is the function defined by the sum, difference, product and quotient of the two formulas, however, the domain is not the maximal domain implied by the resulting formula, but the intersection of the domains of the original functions, or in case of the quotient, an even more restricted set.

For example if $f(x) := x - \sqrt{x}$ and $g(x) := x + \sqrt{x}$ then $D_f = D_g = [0, \infty]$, and thus

$$(f + g) : [0, \infty] \to \mathbb{R}, \quad (f + g)(x) = 2x,$$

even if the function defined simply by the formula $2x$ would have maximal domain $\mathbb{R}$.

Example. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions given by $f(x) := x^3 + 3x^2$ and $g(x) := 1 + \cos^2 x$. then we have

$$(f + g)(x) = f(x) + g(x) = x^3 + 3x^2 + 1 + \cos^2 x$$
$$(f - g)(x) = f(x) - g(x) = (x^3 + 3x^2) - (1 + \cos^2 x) = x^3 + 3x^2 - 1 - \cos^2 x$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = (x^3 + 3x^2) \cdot (1 + \cos^2 x) = x^3 + 3x^2 + x^3 \cos^2 x + 3x^2 \cos^2 x$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^3 + 3x^2}{1 + \cos^2 x}$$

Definition 1.16. (Composition of functions) Let $f : B \to C$ and $g : A \to B$ two functions such that $R_g \subset D_f$, where R_g denotes the range of g and D_f the domain of f. Then the composite function $f \circ g$ is defined as

$$f \circ g : A \to C, \quad (f \circ g)(x) = f(g(x))$$

for every x in the domain of g.

Remark. The composition of functions may also be regarded as connecting two machines. Namely, the first machine g transforms the input x to the output $g(x)$, which will be the input of the machine f, and this transforms $g(x)$ to $f(g(x))$. Then the "connected machine" $f \circ g$ is the machine which transforms the input x to $f(g(x))$, and we already cannot see which part of the "work" was done by the machines f and g, respectively.

Example. diagrammos szemleltetes

Remark. In most cases we shall work with composite functions in the case $A = B = C = \mathbb{R}$, so the composite functions $f \circ g$ and $g \circ f$ can be both defined. However, these functions in general are different, i.e. the composition of functions is not commutative.

Example. Let $f(x) := x^3 + 3x^2$, $g(x) := x^2 - 2$ and $h(x) = \cos x$. Then

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) = f(x^2 - 2) = (x^2 - 2)^3 + 3(x^2 - 2)^2 = \\
&= x^6 - 6x^4 + 12x^2 - 8 + 3x^4 - 12x^2 + 12 = x^6 - 3x^4 + 4 \\
(g \circ f)(x) &= g(f(x)) = g(x^3 + 3x^2) = (x^3 + 3x^2)^2 - 2 = x^6 + 6x^5 + 9x^4 - 2 \\
(f \circ h)(x) &= f(h(x)) = f(\cos x) = \cos^3 x + 3\cos^2 x \\
(h \circ f)(x) &= h(f(x)) = h(x^3 + 3x^2) = \cos(x^3 + 3x^2) \\
(g \circ g)(x) &= g(g(x)) = g(x^2 - 2) = (x^2 - 2)^2 - 2 = x^4 - 4x^2 + 4 - 2 = x^4 - 4x^2 + 2 \\
(f \circ h \circ g)(x) &= f(h(g(x))) = f(h(x^2 - 2)) = \\
&= f(\cos(x^2 - 2)) = \cos^3(x^2 - 2) + 3\cos^2(x^2 - 2)
\end{aligned}$$

Exercise 1.5. Compute $f \circ g$, $g \circ f$, $f \circ h$, $h \circ f$, $f \circ h \circ g$ and $g \circ g \circ g$, provided that

a) $f(x) = x^3 + x + 1, \quad g(x) = x^2 + 2x, \quad h(x) = 3x + 2$

b) $f(x) = x^3 + 1, \quad g(x) = x^3 - 1, \quad h(x) = \sin x$

c) $f(x) = x^2 + x + 1, \quad g(x) = 2^x, \quad h(x) = \tan x$

d) $f(x) = x^2 + x, \quad g(x) = \sqrt{2x}, \quad h(x) = x - 2$

e) $f(x) = x^3, \quad g(x) = 3^x, \quad h(x) = 3x + 2$

f) $f(x) = x^4, \quad g(x) = \sin x, \quad h(x) = \log_2 x$

g) $f(x) = x^3 - x + 1, \quad g(x) = x^2 - 2, \quad h(x) = 2x + 1$

h) $f(x) = x^3 - 2x + 1, \quad g(x) = x^2 - x + 1, \quad h(x) = x^2 + 1$

i) $f(x) = x^2 - x + 3, \quad g(x) = x^3 + 1, \quad h(x) = x^2 + x + 1$

j) $f(x) = x^2 - 2x + 2, \quad g(x) = x^2 + 3, \quad h(x) = x^3 + x$

k) $f(x) = x^3 - 2x, \quad g(x) = x^2 + 1, \quad h(x) = x^2 + x$

l) $f(x) = x^3 + x + 1, \quad g(x) = x^2 - x - 1, \quad h(x) = x^2 + x + 1$

Example. Let $f_k(x) := x^k$ and $g_k(x) := \sqrt[k]{x}$ for $k = 1, 2, \ldots$. Let $h_a(x) := a^x$ and $l_a(x) := \log_a(x)$ for $\in \mathbb{R}_{>0}$ and $a \neq 1$. Further, let $s(x) := \sin x$, $c(x) = \cos x$ and $t(x) := \tan x$. Build the following function $F(x)$ as a composite function of the above functions:

$$F(x) := \sin^3 \left(\sqrt[5]{\log_2 (sin(2^x))} \right).$$

Solution.

$$F(x) = f_3 (s (g_5 (l_2 (s (h_2(x))))))$$

Exercise 1.6. Let $f_k(x) := x^k$ and $g_k(x) := \sqrt[k]{x}$ for $k = 1, 2, \ldots$. Let $h_a(x) := a^x$ and $l_a(x) := \log_a(x)$ for $\in \mathbb{R}_{>0}$ and $a \neq 1$. Further, let $s(x) := \sin x$, $c(x) = \cos x$ and $t(x) := \tan x$. Build the following function $F(x)$ as a composite function of the above functions:

a) $F(x) := \cos^4 \left(\sqrt[7]{\log_3 (\tan(5^x))} \right)$

b) $F(x) := \log_3^7 \left(\sin^2 \sqrt[7]{\log_3} (\tan(5^x)) \right)$

c) $F(x) := \tan \left(\log_2 \left(\cos \left(\sqrt[4]{\log_5 (\sin(3^x))} \right) \right) \right)^4$

d) $F(x) := \sin^3 \left(\log_2 \tan \sqrt{7^{\sin x^5}} \right)^4$

e) $F(x) := \log_3^5 \left(\sin^2 \sqrt[6]{\cos 11^{\log_4 x}} \right)$

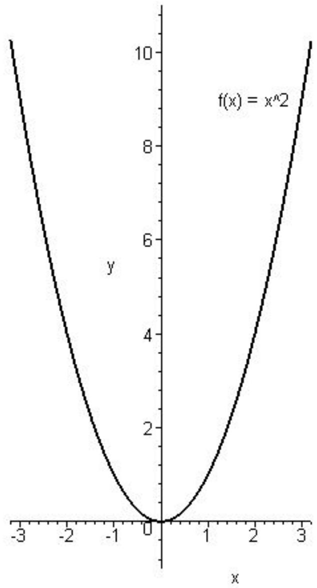
f) $F(x) := \sin^2 \tan^5 \log_2^3 \cos \left(\sqrt{7^{x^2}} \right)$

Definition 1.17. Let A, B and $X \subseteq A$ be sets, and let $f : A \to B$ be a function. The function $g : X \to B$, for which $g(x) = f(x)$ for each $x \in X$, is said to be the *the restriction of f to the set X*.

The restriction of the function f to the set X is also denoted by $f|_X$.

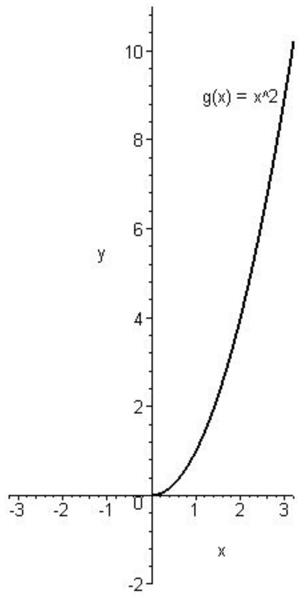
Examples.

1. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. The graph of f is:



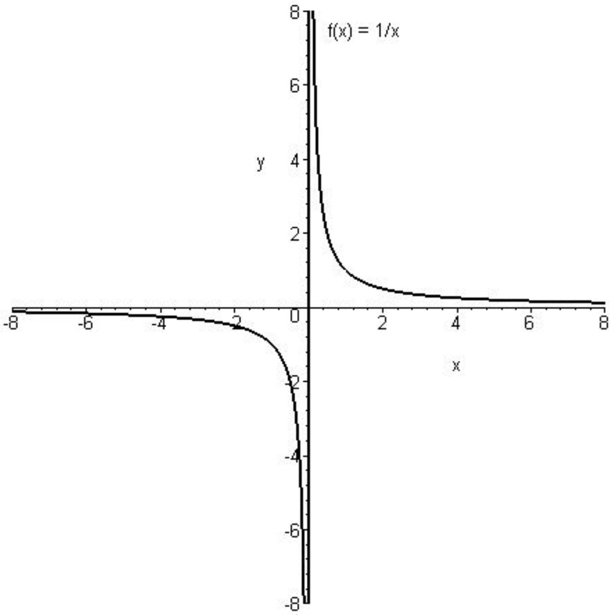
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

The restriction g of f to the set $[0, \infty[$ is $g : [0, \infty[\to \mathbb{R}$, $g(x) = x^2$. The graph of g is:



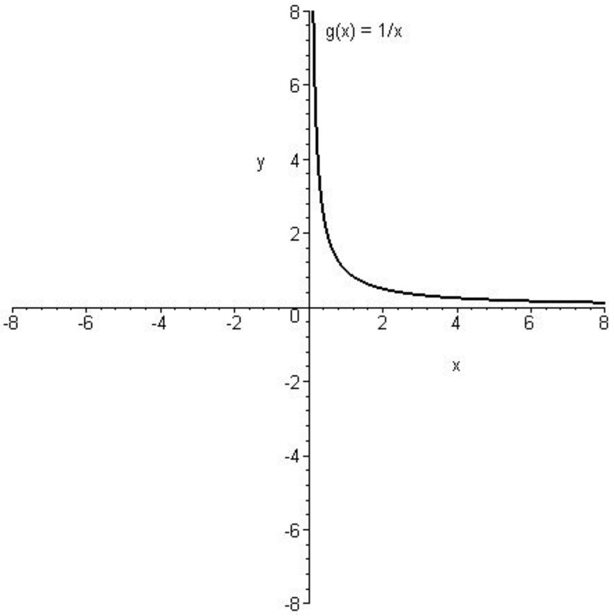
Graph of the function $g : [0, \infty[\to \mathbb{R}$, $g(x) = x^2$.

2. Let $f : \mathbb{R} \setminus 0 \to \mathbb{R}$, $f(x) = \frac{1}{x}$. The graph of f is:



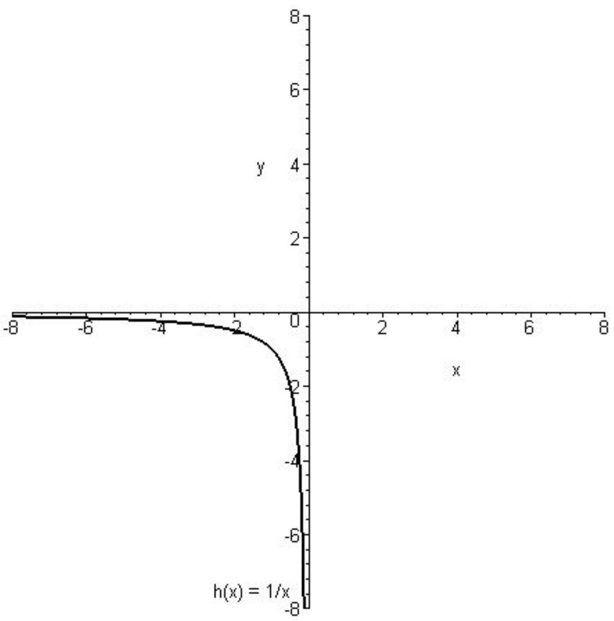
Graph of the function $f : \mathbb{R} \setminus 0 \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

The restriction g of f to the set $]0, \infty[$ is $g :]0, \infty[\to \mathbb{R}$, $g(x) = \frac{1}{x}$. The graph of g is:



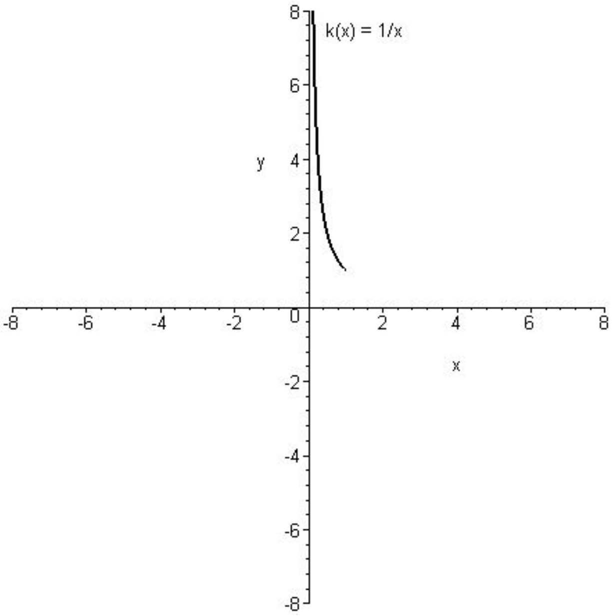
Graph of the function $g :]0, \infty[\to \mathbb{R}$, $g(x) = \frac{1}{x}$.

The restriction h of f to the set $]-\infty, 0[$ is $h :]\infty, 0[\to \mathbb{R}$, $h(x) = \frac{1}{x}$. The graph of h is:



Graph of the function $h :]0, \infty[\to \mathbb{R}$, $h(x) = \frac{1}{x}$.

The restriction k of f to the set $]0, 1]$ is $k :]0, 1] \to \mathbb{R}$, $k(x) = \frac{1}{x}$. The graph of k is:



Graph of the function $k :]0, 1] \to \mathbb{R}$, $k(x) = \frac{1}{x}$.

1.5 Further properties of functions

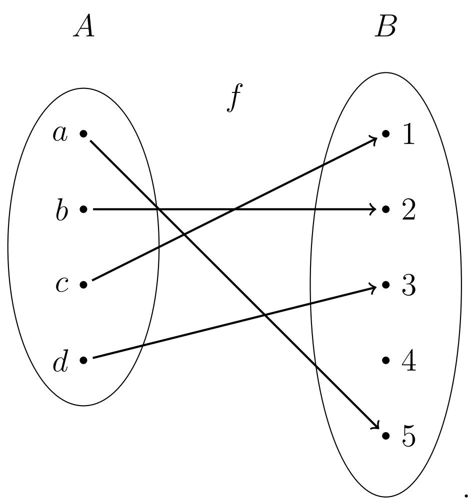
1.5.1 Injectivity of functions

Definition 1.18. Let A and B arbitrary sets. A function $f : A \to B$ is called *injective* if, for all $x, z \in A$, $y \in B$ for which $(x, y) \in f$ and $(z, y) \in f$, we have $x = z$.

Remark. Injective functions are also called *invertible*.

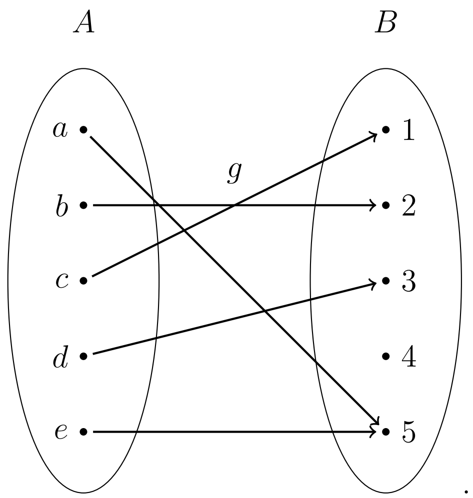
Examples.

1. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$ and let us define $f : A \to B$ such that $f(a) = 5$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$. It is easy to see that f satisfies the requirements given in Definition 1.6, therefore, it is a function. It can be illustrated in the following form:



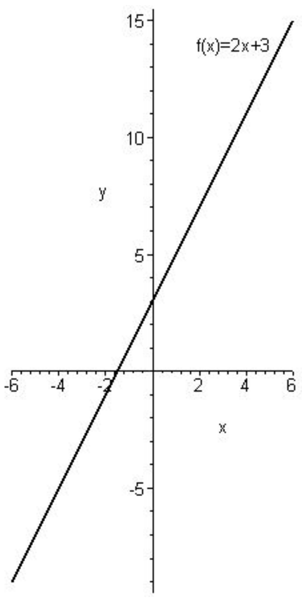
   It is visible, that there are no elements $x_1 \in A$ and $x_2 \in A$ such that $f(x_1) = f(x_2)$, which implies that f is injective.

2. Let now $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $g : A \to B$, $g(a) = 5$, $g(b) = 2$, $g(c) = 1$, $g(d) = 3, g(e) = 5$. This function can be illustrated as



It is easy to see that g is a function, but, since $g(a) = g(e) = 5$, it is not injective.

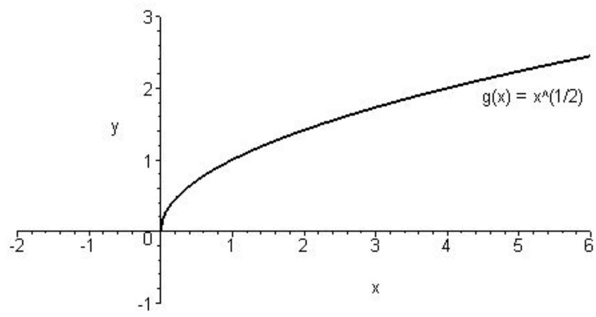
3. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$. The graph of this function has the following form.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$.

It is easy to see, that there are no elements $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$, which yields the injectivity of f.

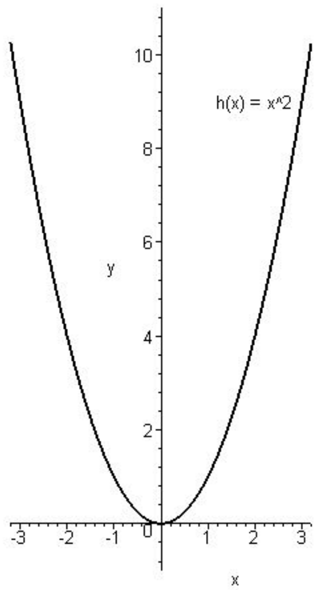
4. Let $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x}$. The graph of g is:



Graph of the function $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x}$.

Obviously, there no no elements $x_1 \in [0, \infty[$ and $x_2 \in [0, \infty[$ such that $g(x_1) = g(x_2)$, thus g is also injective.

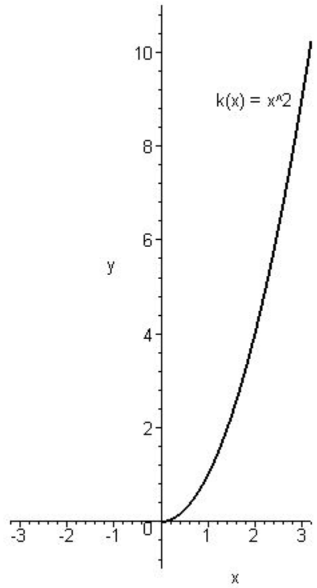
5. Let $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2$. The graph of h:



Graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2$.

Since there are elements $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$ such that $h(x_1) = h(x_2)$ (for example $f(-1) = f(1) = 1$), the function h is not injective.

6. Let us consider the function $k : [0, \infty[\to \mathbb{R}$, $k(x) = x^2$ now. The graph of k is:



Graph of the function $k : [0, \infty[\to \mathbb{R}$, $k(x) = x^2$.

Obviously, there does not exist elements $x_1 \in [0, \infty[$ and $x_2 \in [0, \infty[$ such that $k(x_1) = k(x_2)$, therefore k is an injective function.

Remark. The last two examples point out that if we investigate the injectivity of a function then the domain of the function plays an important role.

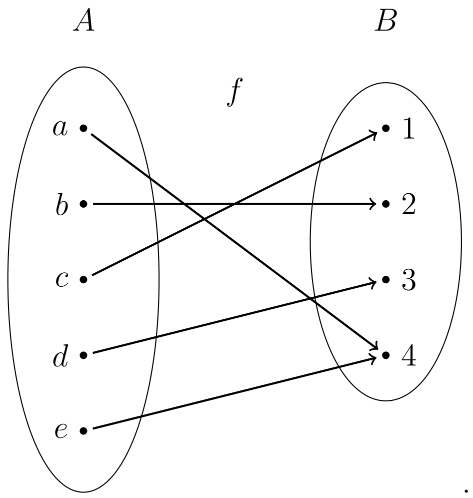
1.5.2 Surjectivity of functions

Definition 1.19. Let A and B arbitrary sets. A function $f : A \to B$ is said to be *surjective* if the range of f is B.

Remark. The definition above requires that, for each $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

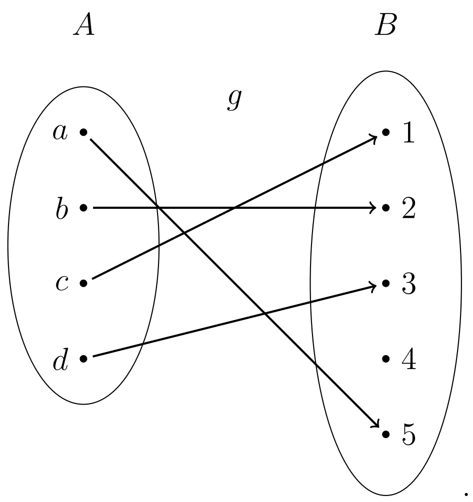
Examples.

1. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$ such that $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$, $f(e) = 4$, i.e.,



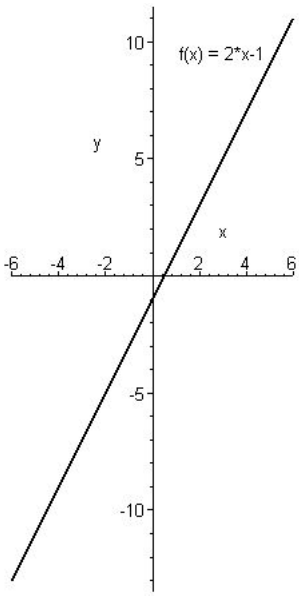
It is visible that f is a function, furthermore, it is also clear that, for each element $y \in B$, there exists an $x \in A$ such that $f(x) = y$, therefore f is surjective.

2. Let now $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$ and let $g : A \to B$, $g(a) = 5$, $g(b) = 2$, $g(c) = 1$, $g(d) = 3$. A diagram corresponding to g is:



It is easy to see that g is a function. However, there exists an element $y \in B$ such that there is is no $x \in A$ for which $g(x) = y$. (The element $4 \in B$ has this property.) This implies that g is not surjective.

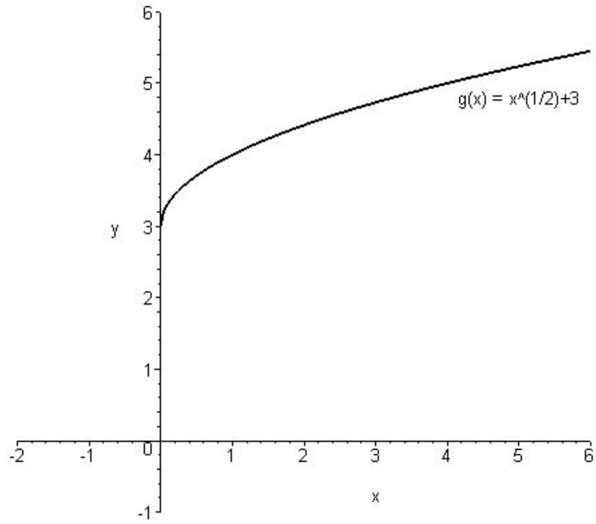
3. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 1$. The graph of this function has the following form.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 1$.

It is easy to verify that, for each $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$ for which $f(x) = y$. (Solving the equation $y = 2x - 1$ for x, we obtain such an $x \in \mathbb{R}$ for each $y \in R$.) This means that the function f is surjective.

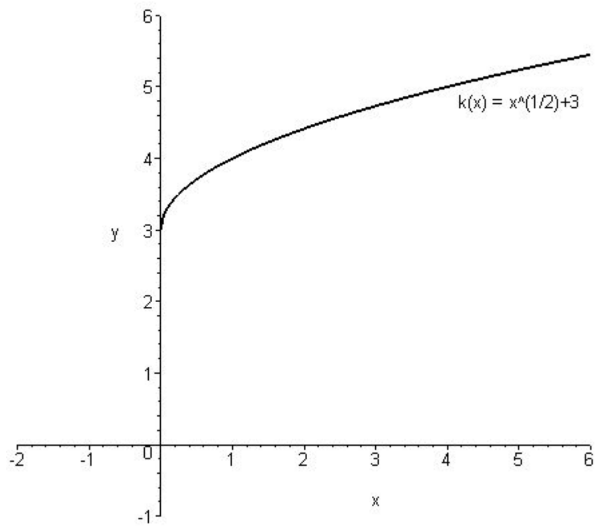
4. Let $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x} + 3$. The graph of g is:



Graph of the function $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x} + 3 = x^{1/2} + 3$.

Since there are elements y in the range $\mathbb{R}$ of g such that there is no element x in the domain $[0, \infty[$ of g for which $g(x) = y$. (For example $y = 1$ fulfills this property.)) Therefore g is not surjective.

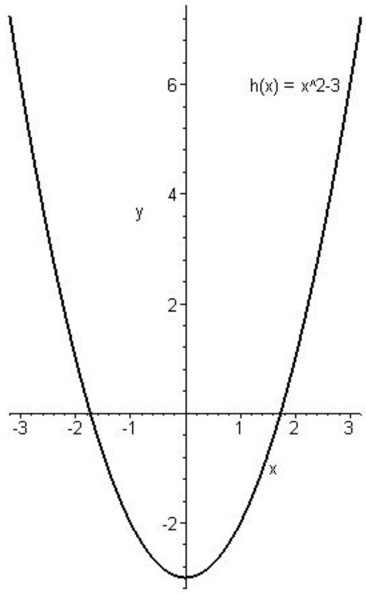
5. Let $k : [0, \infty[\to [3, \infty[$, $k(x) = \sqrt{x} + 3$. The graph of g is:



Graph of the function $k : [0, \infty[\to [3, \infty[$, $k(x) = \sqrt{x} + 3 = x^{1/2} + 3$.

It is easy to see that, for each $y \in [3, \infty[$ there exists an $x \in [0, \infty[$ such that $k(x) = y$ Thus, k is surjective.

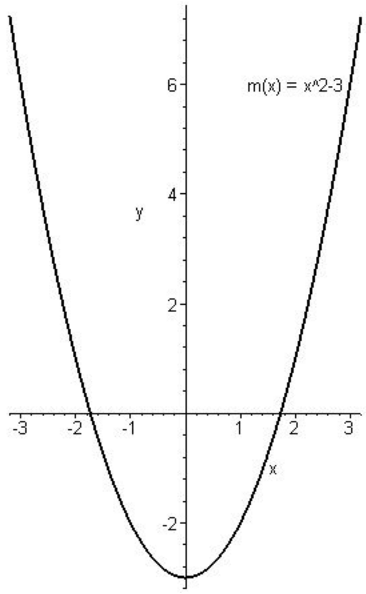
6. Let $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 - 3$. The graph of h:



Graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^2 - 3$.

It is easy to check that h is not surjective.

7. Let $m : \mathbb{R} \to [-3, \infty[$, $m(x) = x^2 - 3$. The graph of m:



Graph of the function $m : \mathbb{R} \to [-3, \infty[$, $m(x) = x^2 - 3$.

It is obvious that m is surjective.

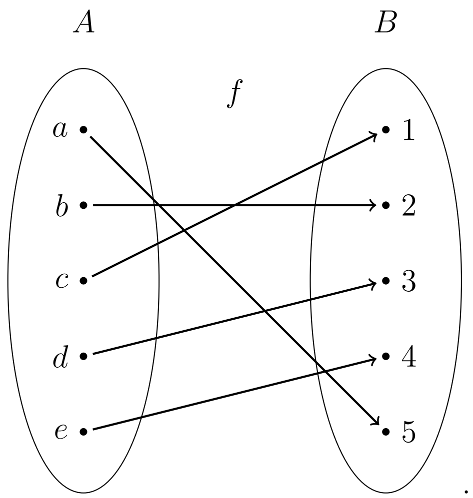
Remark. The last four examples show that concerning the injectivity of a function the range of the function plays an important role.

1.5.3 Bijectivity of functions

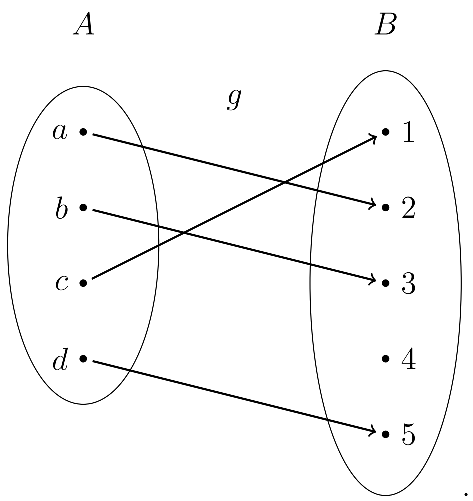
Definition 1.20. Let A and B arbitrary sets. A function $f : A \to B$ is called *bijective* if it is injective and surjective.

Examples.

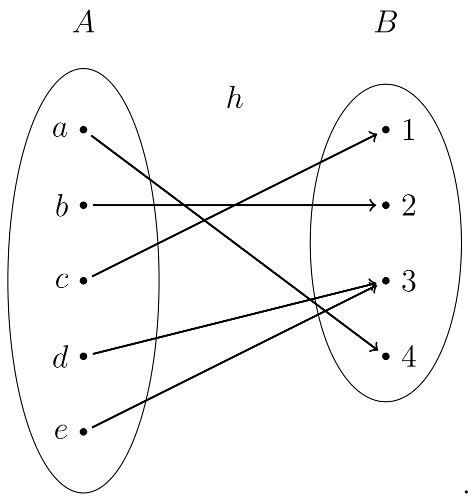
1. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 5$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$, $f(e) = 4$. It is easy to see that f is a function, furthermore, it is injective and surjective, therefore, it is bijective. The function has the form



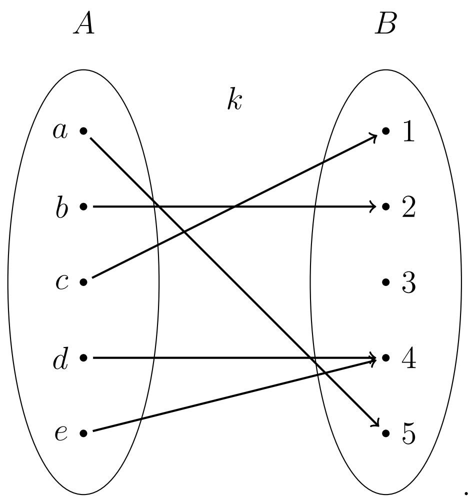
2. Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$ and let $g : A \to B$, $g(a) = 2$, $g(b) = 3$, $g(c) = 1$, $g(d) = 4$. This function is injective but not surjective, thus, it is not bijective. An illustration of g is:



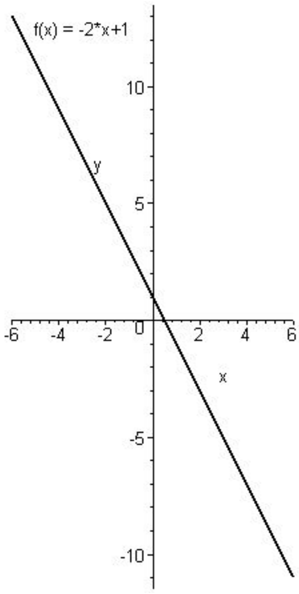
3. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $h : A \to B$, $h(a) = 5$, $h(b) = 2$, $h(c) = 1$, $h(d) = 3, h(e) = 3$. The function h is surjective but not injective, which yields that it is not bijective. The function can be illustrated in the following form::



4. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $k : A \to B$, $k(a) = 5$, $k(b) = 2$, $k(c) = 1$, $k(d) = 4, k(e) = 4$. The function k is neither injective nor surjective, therefore, it is not bijective. The function has the form



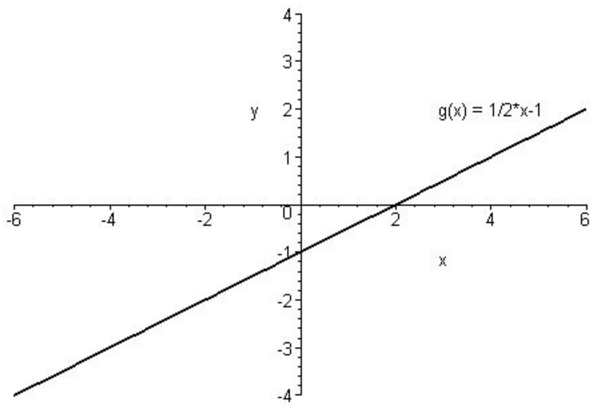
5. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x + 1$. The graph of this function has the form:



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x + 1$.

It is easy to see that this function is injective and surjective, therefore, it is bijective.

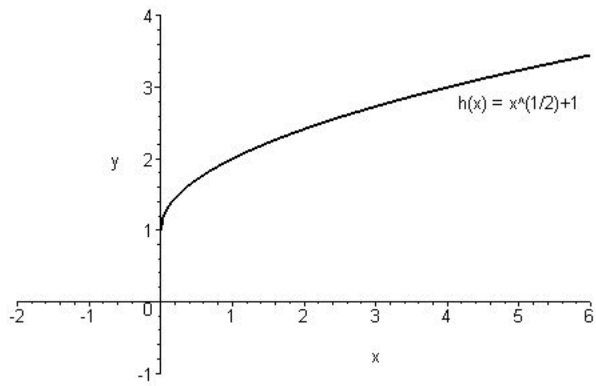
6. Let us consider the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x - 1$. The graph of g is:



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x - 1$.

This function is also injective and surjective, therefore, it is bijective.

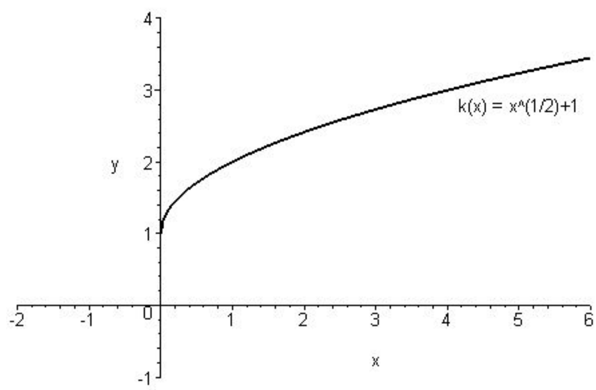
7. Let $h : [0, \infty[\to \mathbb{R}$, $h(x) = \sqrt{x} + 1$. The graph of h is:



Graph of the function $h : [0, \infty[\to \mathbb{R}$, $h(x) = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$.

The function h is injective but not surjective, thus, it is not bijective.

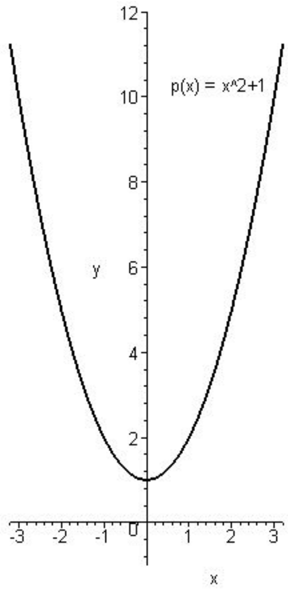
8. Let $k : [0, \infty[\to [1, \infty[$, $k(x) = \sqrt{x} + 1$. The graph of k is:



Graph of the function $k : [0, \infty[\to [1, \infty[$, $k(x) = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$.

The function k is injective and surjective, therefore, it is bijective.

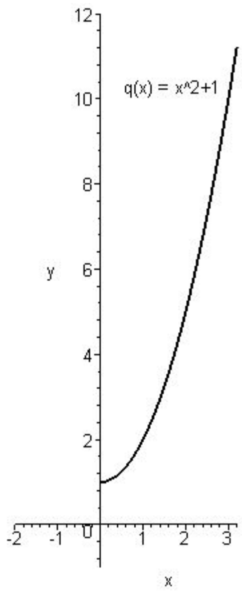
9. Let $p : \mathbb{R} \to \mathbb{R}$, $p(x) = x^2 + 1$. The graph of p:



Graph of the function $p : \mathbb{R} \to \mathbb{R}$, $p(x) = x^2 + 1$.

This function is neither injective nor surjective, thus, it is not bijective.

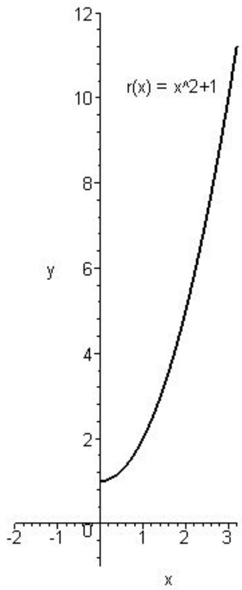
10. Let $q : [0, \infty[\to \mathbb{R}$, $q(x) = x^2 + 1$. The graph of q:



Graph of the function $q : [0, \infty[\to \mathbb{R},\ q(x) = x^2 + 1$.

The function q is injective but not surjective, therefore, it is not bijective.

11. Let $r : [0, \infty[\to [1, \infty[\,,\ r(x) = x^2 + 1$. The graph of r:



Graph of the function $r : [0, \infty[\to [1, \infty[\,,\ r(x) = x^2 + 1$.

The function r is injective and surjective, thus, it is bijective.

Remark. The examples above show that concerning the bijectivity of a function its domain and its range is also important.

Exercise 1.7. Illustrate the following functions and decide whether they are bijective, injective or surjective:

a) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 5$, $f(e) = 2$;

b) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 5$, $f(e) = 3$;

c) $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 2$, $f(d) = 5$;

d) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 2$, $f(e) = 2$;

e) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 2$, $f(e) = 3$;

$A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 4$, $f(c) = 3$, $f(d) = 1$, $f(e) = 2$.

Exercise 1.8. Draw the graph of the following functions in a coordinate system and decide whether they are bijective, injective or surjective:

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$;

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3x + 2$;

c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x + 3$;

d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2$;

e) $f : \mathbb{R} \to [2, \infty[$, $f(x) = x^2 + 2$;

f) $f : [0, \infty[\to [2, \infty[$, $f(x) = x^2 + 2$;

g) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$;

h) $f : \mathbb{R} \to [-1, \infty[$, $f(x) = x^2 - 1$;

i) $f : [0, \infty[\to [-1, \infty[$, $f(x) = x^2 - 1$;

j) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (x - 2)^2$;

k) $f : \mathbb{R} \to [0, \infty[$, $f(x) = (x - 2)^2$;

l) $f : [2, \infty[\to [0, \infty[$, $f(x) = (x - 2)^2$;

1.5.4 Inverse of a function

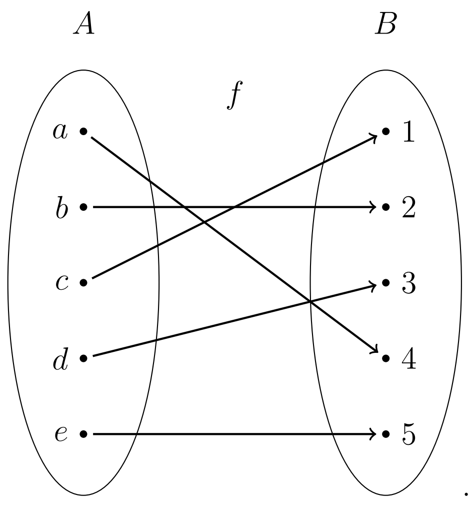
Definition 1.21. Let A and B arbitrary sets, let $f : A \to B$ be a injective function and let us denote the range of f by Y. Interchanging the elements in the ordered pairs $(x, y) \in f$, we obtain a function $g : Y \to A$, $g(y) = x$. This function is said to be the *inverse function* of f.

The inverse function of f is denoted by f^{-1}.

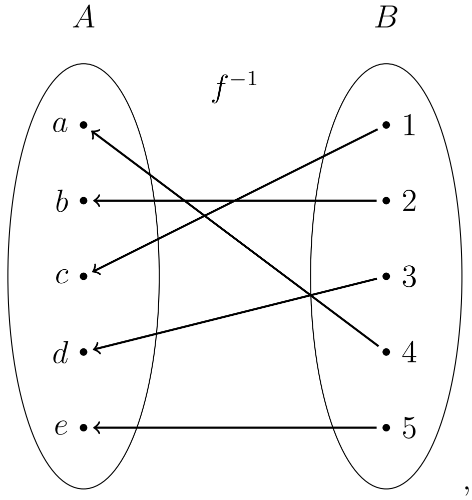
Remark. It is important to note that the inverse function is defined in the case of injective functions only.

Examples.

1. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let us define $f : A \to B$, $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, $f(d) = 3$, $f(e) = 5$. This function can be illustrated as:

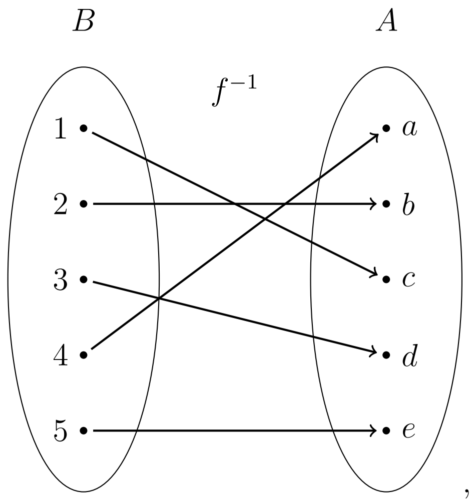


It is easy to see that this function is injective, thus there exists its inverse. The inverse $f^{-1} : B \to A$ of f has the form



,

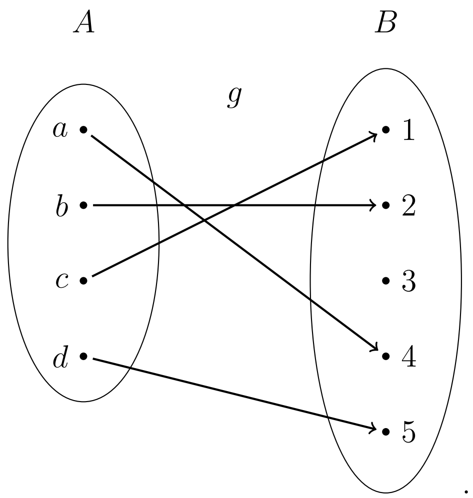
that is,



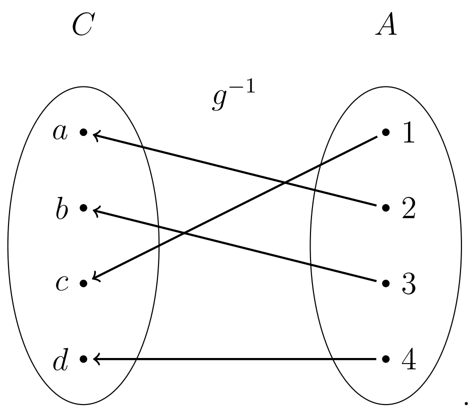
,

which means that $f^{-1}(1) = c$, $f^{-1}(2) = b$, $f^{-1}(3) = d$, $f^{-1}(4) = a$ and $f^{-1}(5) = e$.

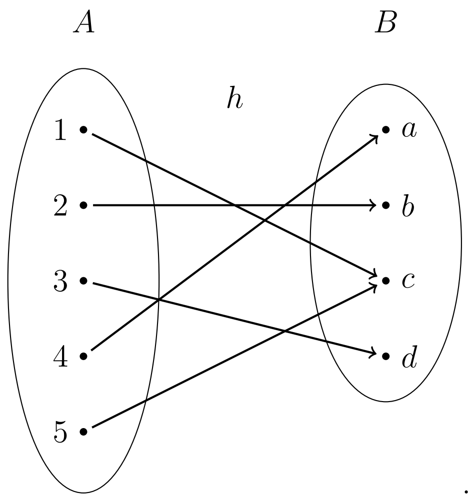
2. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let us define $g : A \to B$, $g(a) = 4$, $g(b) = 2$, $g(c) = 1$, $g(d) = 5$. This function can be illustrated as:



The function g is injective, therefore, it is invertible. It is important to mention, that the range of g is the set $C = \{1, 2, 3, 5\}$, which means that the domain of g^{-1} will be this set, too. Thus, $g^{-1} : C \to A$, $g^{-1}(1) = c$, $g^{-1}(2) = a$, $g^{-1}(3) = b$, $g^{-1}(5) = d$, that is,

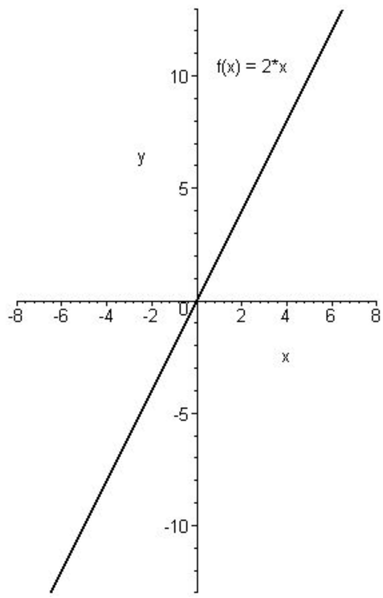


3. Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$ and let $h : A \to B$, $h(a) = 4$, $h(b) = 2$, $h(c) = 1$, $h(d) = 3, h(e) = 3$, i. e.,



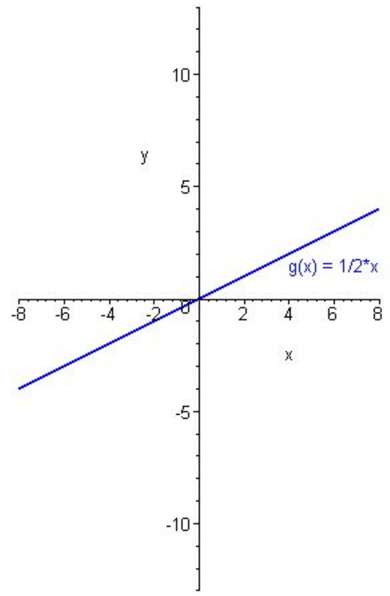
The function h is not injective, thus, it does not have any inverse.

4. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. The graph of this function has the form:



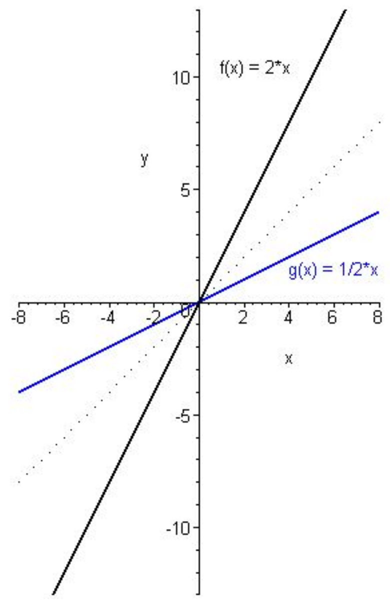
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$.

This function is injective, therefore, it is invertible. Its inverse can be determined in the following form: by the definition of f, we may write $y = 2x$ for all $x \in \mathbb{R}$, which means that $x = \frac{1}{2}y$ for each $y \in \mathbb{R}$. Therefore, the inverse $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ of f can be written as $g(y) = f^{-1}(y) = \frac{1}{2}y$ or, writing x instead of y, $g(x) = f^{-1}(x) = \frac{1}{2}x$. The graph of $g = f^{-1}$ is:



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x$.

Let us draw the graphs of the functions f and $g = f^{-1}$ above in the same coordinate system.



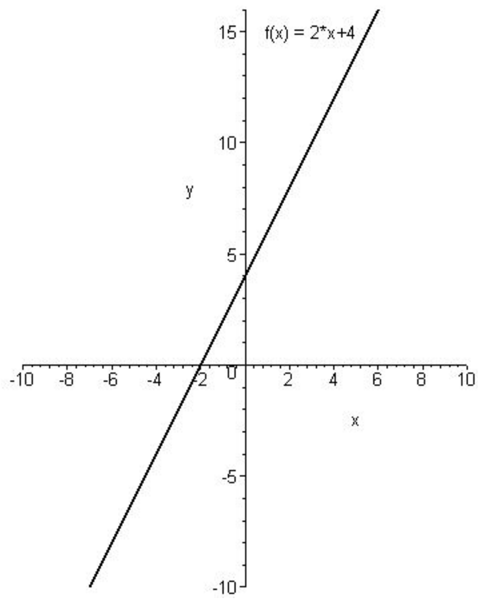
Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ and its inverse $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x$.

We may observe that the graph of $g = f^{-1}$ is a reflection of the graph of f in the line $y = x$. (More precisely the graph of $g = f^{-1}$ is a reflection of the graph of f in the graph of the identical function $i : \mathbb{R} \to \mathbb{R}$, $i(x) = x$.) This important connection is always valid for the graph of a function and for the graph of its inverse. We will formulate this fact in the next Remark.

Remark. It is easy to verify that the graph of the inverse of an invertible function f is always a reflection of the graph of f in the graph of the identical function $i : \mathbb{R} \to \mathbb{R}$, $i(x) = x$.

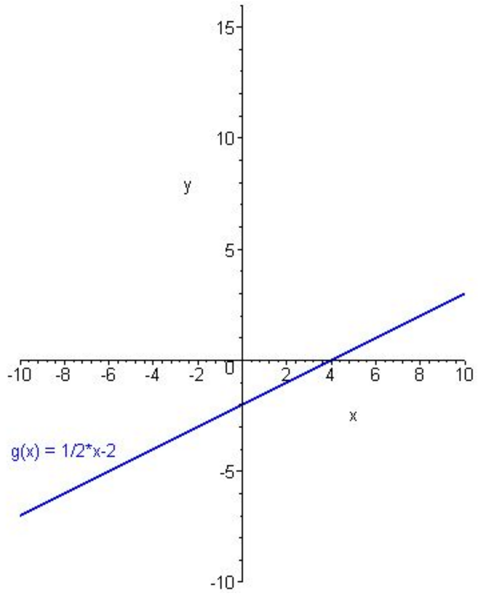
Examples.

1. Let us consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$ now.



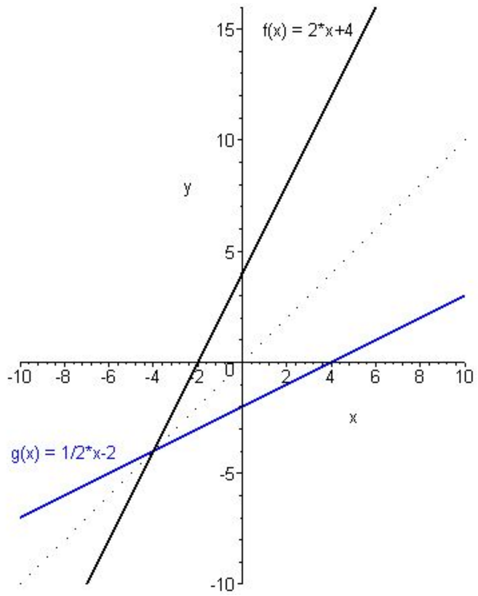
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$.

This function is injective, thus, it is invertible. The equation $y = 2x + 4$ implies that $x = \frac{1}{2}y - 2$, therefore the inverse $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ of f has the form $g(x) = \frac{1}{2}x - 2$.



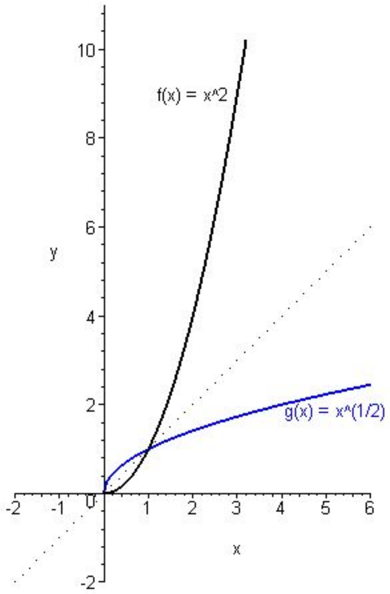
Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x - 2$.

Drawing the graphs of f and g in the same coordinate system, we obtain:



Graphs of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$ and its inverse $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \frac{1}{2}x - 2$.

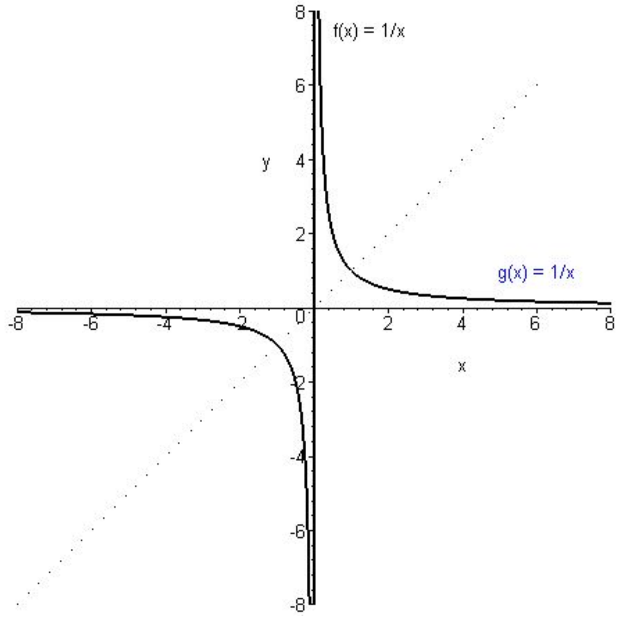
2. Let $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2$. This function is injective, therefore, it is invertible. Its inverse is is $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x}$. The graphs of these functions are:



Graphs of $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2$ and its inverse $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{x}$.

3. Note that the function $\mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not injective, thus, its inverse does not exist.

4. Let $f : \mathbb{R} \setminus 0 \to \mathbb{R}$, $f(x) = \frac{1}{x}$. This function is injective, thus, invertible. It inverse is $g : \mathbb{R} \setminus 0 \to \mathbb{R}$, $f(x) = \frac{1}{x}$, that is the same function as f The graph of the functions



Graphs of $f : \mathbb{R} \setminus 0 \to \mathbb{R}$, $f(x) = \frac{1}{x}$ and its inverse $g : \mathbb{R} \setminus 0 \to \mathbb{R}$, $g(x) = \frac{1}{x}$.

Exercise 1.9. Illustrate the following functions with a diagram, investigate if they are invertible or not and, in the case when they exist, determine their inverse functions and draw their diagrams:

a) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 5$, $f(b) = 3$, $f(c) = 1$, $f(d) = 2$, $f(e) = 4$;

b) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$,
$f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 2$, $f(d) = 1$;

c) $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$,
$f : A \to B$, $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, $f(d) = 5$;

d) $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$,
$f : A \to B$, $f(a) = 1$, $f(b) = 3$, $f(c) = 4$, $f(d) = 2$, $f(e) = 3$.

Exercise 1.10. Draw the graphs of the following functions, determine their inverses (if they exist) and draw the graph of the inverse functions, too:

a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$;

b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 4$;

c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$;

d) $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2 + 1$;

e) $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2 - 2$;

f) $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} + 1$;

g) $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} - 2$.

Chapter 2

Types of functions

2.1 Linear functions

Definition 2.1. A function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := ax + b, \quad a \in \mathbb{R}, b \in \mathbb{R}$$

is called a *linear function*.

Remark. We remind the student that for all pairs of points $P_(x_1, y_1)$ and $P_2(x_2, y_2)$ lying on the same (not vertical) straight line in the plane (represented in a Cartesian coordinate system) the quantity

$$\frac{y_2 - y_1}{x_2 - x_1} := a$$

is constant (i.e. it is independent of the choice of P_1 and P_2). This number is called the slope of the line.

Theorem 2.2. *Let $f(x) = ax + b$ be a linear function with $a, b \in \mathbb{R}$*

1). **The graph of f is a straight line.** *More precisely, the graph of f is the line with equation $y = ax + b$. Further, the graph of f is a horizontal line if and only if $a = 0$.*

2). The Y-intercept of the graph of f is the point $B(0, b)$.

3). Whenever $a \neq 0$ the only X-intercept of the graph of f is the point $A\left(\frac{-b}{a}, 0\right)$. If $a = 0, b \neq 0$ then the graph of f has no X-intercept. If $a = 0, b = 0$ then the graph of f is the X-axis itself.

The above theorem suggests the definition below:

Definition 2.3. The number a is called the slope of the linear function $f(x) = ax + b$.

Theorem 2.4. *Fix a Cartesian coordinate system in the two dimensional plane. Except for the vertical lines every straight line in the plane is the graph of a linear function. The function corresponding to a line may be given by the following formulas:*

1). **The slope-intercept formula**

> *If the line has slope a and Y-intercept b then the function corresponding to the line is given by*
> $$f(x) = ax + b.$$

2). **The point-slope formula**

> *If the line has slope a and passes through the point $P_1(x_1, y_1)$ then the line has equation*
> $$y - y_1 = a(x - x_1),$$
> *and the function corresponding to the line is given by*
> $$f(x) = a(x - x_1) + y_1.$$

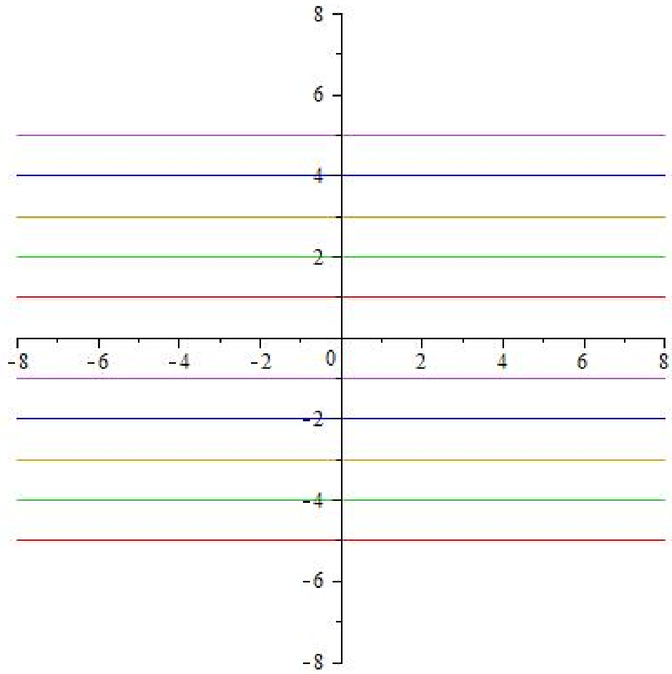
3). **The point-point formula**

> *If the passes through the points $P_1(x_1, y_1)$ and $P_1(x_1, y_1)$ with $y_1 \neq y_2$ then the line has equation*
> $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$
> *and the function corresponding to the line is given by*
> $$f(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

Example. Draw the graph of the following functions:

$$f_b(x) = b, \quad b = -4, -3, -2, -1, 0, 1, 2, 3, 4.$$

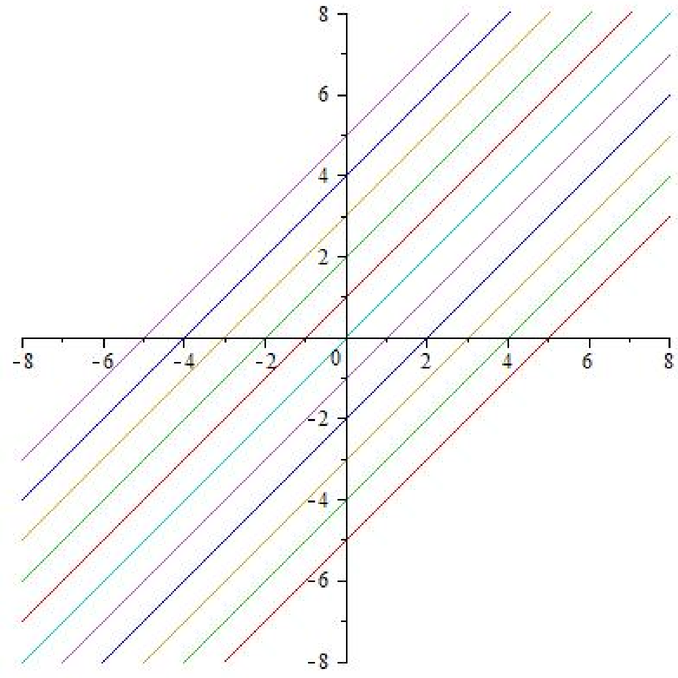
As the below figure shows the graph of these functions are all horizontal lines and the intersection with the Y-axis is b.



Example. Draw the graph of the following functions:

$$f_b(x) = x + b, \quad b = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5.$$

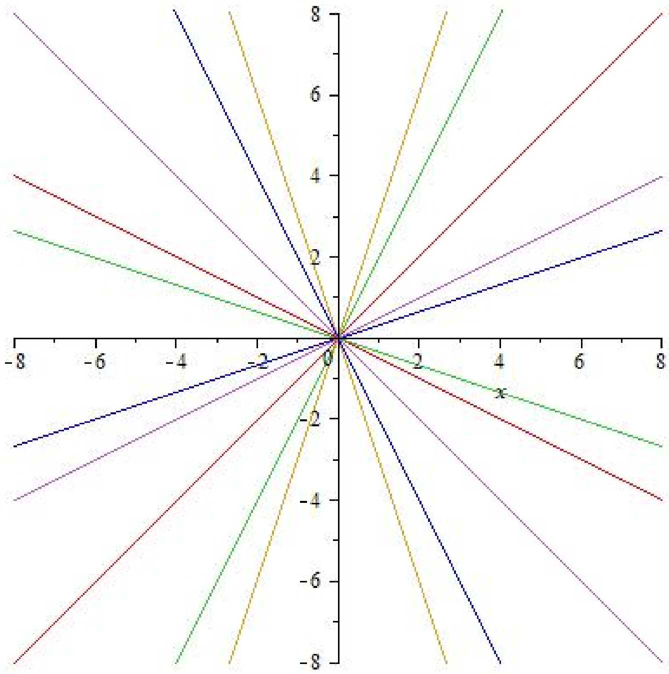
As the below figure shows the graph of these functions are parallel to each other and the intersection with the Y-axis is b.



Example. Draw the graph of the following functions:

$$f_a(x) = ax, \quad a = -3, -2, -1, -0.5, -1/3, 0, 1/3, 0.5, 1, 2, 3, 4, 5.$$

As the below figure shows the graph of each of these functions goes through the origin, and if a is positive, then the line passes through the 1st and 3rd quadrant, if a is negative, then it passes through the 2nd and 4th quadrant. In the case of $a = 0$ the graph is the X-axis



Example. Determine the linear function whose slope is $a = 3$ and whose graph has Y-intercept $b = -5$.

Solution. By the slope-intercept formula

$$f(x) = ax + b = 3x - 5.$$

Example. Determine the linear function whose slope is $a = 3$ and whose graph passes through the point $P_1(-2, 4)$.

Solution. By the point-slope formula the equation of the line is

$$\begin{aligned}(y - y_1) &= a(x - x_1)\\ y - 4 &= 3(x - (-2))\\ y &= 3x + 10\end{aligned}$$

and thus the linear function whose graph is this line is given by

$$f(x) = 3x + 10.$$

Example. Determine the linear function whose graph passes through the points $P_1(-3, 1)$ and $P_2(2, 11)$.

Solution. By the point-point formula the equation of the line is

$$\begin{aligned}
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \\
y - 1 &= \frac{11 - 1}{2 - (-3)}(x - (-3)) \\
y - 1 &= 2(x + 3) \\
y &= 2x + 7
\end{aligned}$$

and thus the linear function whose graph is this line is given by

$$f(x) = 2x + 7.$$

Exercise 2.1. Determine the linear function whose slope is a and whose graph has Y-intercept b where:

a) $a = 3,\ b = -2$ b) $a = 4,\ b = 1$
c) $a = -2,\ b = 5$ d) $a = -5,\ b = -4$

Sketch the graph of these functions.

Exercise 2.2. Determine the linear function whose slope is a and whose graph passes through the point P_1.

a) $a = 2,\ P_1(2, 3)$ b) $a = 3,\ P_1(-2, -4)$
c) $a = -1,\ P_1(3, -3)$ d) $a = 3,\ P_1(-1, 4)$
e) $a = 0,\ P_1(\sqrt{2}, 4)$ f) $a = -2,\ P_1(2, 5)$
g) $a = -7,\ P_1(1, -5)$ h) $a = 5,\ P_1(2, 6)$
i) $a = -3,\ P_1(2, 1)$ j) $a = 4,\ P_1(-3, 10)$
k) $a = -3,\ P_1(2, 5)$ l) $a = 4,\ P_1(3, 19)$
m) $a = \sqrt{2},\ P_1(3, 3\sqrt{2} + \sqrt{3})$ n) $a = 2,\ P_1(\sqrt{5}, \sqrt{5})$

Exercise 2.3. Determine the linear function whose graph passes through the points P_1

and P_2

a) $P_1(-2,1),\ P_2(2,7)$

b) $P_1(2,3),\ P_2(5,0)$

c) $P_1(2,-1),\ P_2(4,7)$

d) $P_1(2,11),\ P_2(-1,-4)$

e) $P_1(-1,10),\ P_2(2,-11)$

f) $P_1(2,3),\ P_2(-2,11)$

g) $P_1(2,1),\ P_2(-1,16)$

h) $P_1(2,-12),\ P_2(-1,12)$

i) $P_1(2,15),\ P_2(-3,-40)$

j) $P_1(-2,31),\ P_2(4,-1)$

k) $P_1(-2,1),\ P_2(-5,-8)$

l) $P_1(5,-4),\ P_2(-2,3)$

m) $P_1(2,4),\ P_2(7,9)$

n) $P_1(-7,4),\ P_2(3,4)$

o) $P_1(-4,-12),\ P_2(2,6)$

p) $P_1(-1,7),\ P_2(3,-21)$

q) $P_1(4,-1),\ P_2(-6,-6)$

r) $P_1(3,13),\ P_2(-2,-2)$

s) $P_1(1,\sqrt{3}+\sqrt{2}),\ P_2(-1,\sqrt{3}-\sqrt{2})$

t) $P_1(-\sqrt{2},-4\sqrt{2}),\ P_2(\sqrt{2},2\sqrt{2})$

2.2 Quadratic functions

Definition 2.5. A function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := ax^2 + bx + c, \quad a, b, c \in \mathbb{R}, a \neq 0$$

is called a quadratic function. If the quadratic function is given by a formula of the above shape, we say that it is given in general form.

Theorem 2.6. *(The canonical form of a quadratic function)*

Let $f(x) = ax^2+bx+c$ be a quadratic function with $a, b, c \in \mathbb{R}$, $a \neq 0$. Put $\Delta := b^2-4ac$. Then $f(x)$ can be written in the form

$$f(x) = a(x - x_v)^2 + y_v, \tag{2.1}$$

with $x_v := \frac{-b}{2a}$ and $y_v := \frac{-\Delta}{4a}$.

Proof.

$$\begin{aligned} f(x) &= ax^2 + bx + c = \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = \\ &= a\left(\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) = \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} = \\ &= a\left(x - \frac{-b}{2a}\right)^2 + \frac{-(b^2 - 4ac)}{4a} \end{aligned}$$

□

Example. Compute the canonical form of the following quadratic function:

$$f(x) = 2x^2 - 8x + 3.$$

Solution.

$$\begin{aligned} f(x) =& 2x^2 - 8x + 3 = 2\left(x^2 - 4x + \frac{3}{2}\right) = \\ =& 2\left((x^2 - 4x + 4) - 4 + \frac{3}{2}\right) = \\ =& 2\left((x-2)^2 + \frac{-5}{2}\right) = \\ =& 2(x-2)^2 - 5 \end{aligned}$$

Another solution to the problem is to use directly Theorem 2.6. We have $a = 2, b = -8, c = 3$, and thus $x_v = \frac{-b}{2a} = \frac{-8}{2} = -4$ and $y_v = \frac{-\Delta}{4a} = \frac{-(b^2-4ac)}{4a} = \frac{-(64-4\cdot 2\cdot 3}{8} = -5$, and we get

$$f(x) = a(x - x_v)^2 + y_v = 2(x-2)^2 - 5.$$

Theorem 2.7. *(The factorized form of a quadratic function)*

Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a, b, c \in \mathbb{R}$, $a \neq 0$. Put $\Delta := b^2 - 4ac$ and if $\Delta \geq 0$ then put

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}, \qquad x_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

Then $f(x)$ can be written in the form

$$f(x) = a(x - x_1)(x - x_2). \tag{2.2}$$

If $\Delta = 0$ then $x_1 = x_2$ and the above factorization takes the form $f(x) = a(x - x_1)^2$.

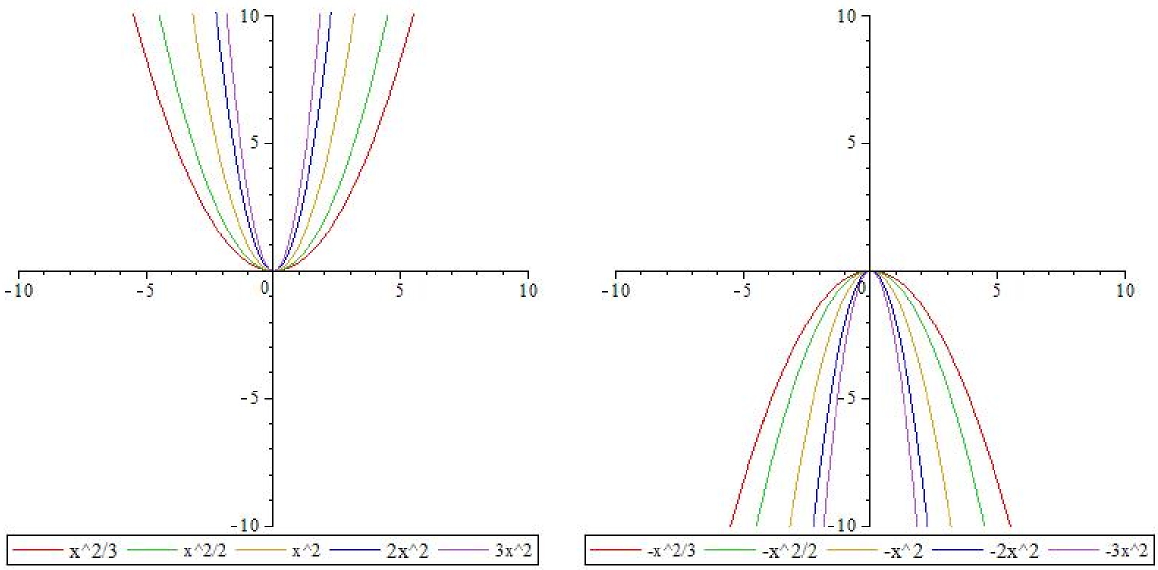
Theorem 2.8. *Let $f(x) = ax^2 + bx + c$ be a quadratic function with $a, b, c \in \mathbb{R}$, $a \neq 0$. Put $\Delta := b^2 - 4ac$ and if $\Delta \geq 0$ then put*

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}, \qquad x_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

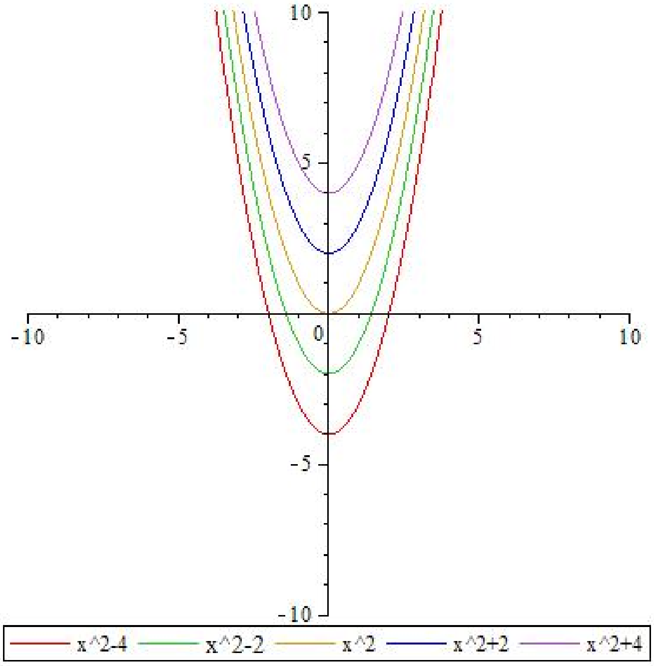
The graph of f is a parabola with the following properties:

- *If $a > 0$ the parabola is "cup-like", if $a < 0$ then the parabola is "bell-like"*
- *The vertex of this parabola is the point $V(x_v, y_v)$ with $x_v := \frac{-b}{2a}$ and $y_v := \frac{-\Delta}{4a}$*
- *The Y-intercept of the graph of f is the point $P(0, c)$*
- *The number of X-intercepts of the graph of f depends on the sign of Δ:*
  - *If $\Delta < 0$ then the graph of f has no X-intercepts;*
  - *If $\Delta = 0$ then $x_1 = x_2 = \frac{-b}{2a}$ and the only X-intercept of the graph of f is the point $A(x_1, 0)$ and this point is just the vertex of the parabola;*
  - *If $\Delta > 0$ then the X-intercepts of the graph of f are the points $A_1(x_1, 0)$ and $A_2(x_2, 0)$.*

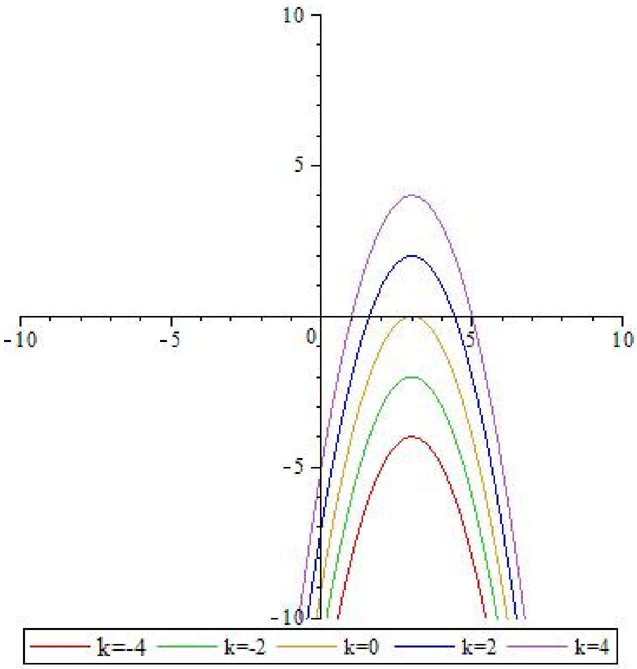
Example. Draw the graph of the functions $f_a(x) := ax^2$ for $a = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$ and for $a = -\frac{1}{3}, -\frac{1}{2}, -1, -2, -3$.



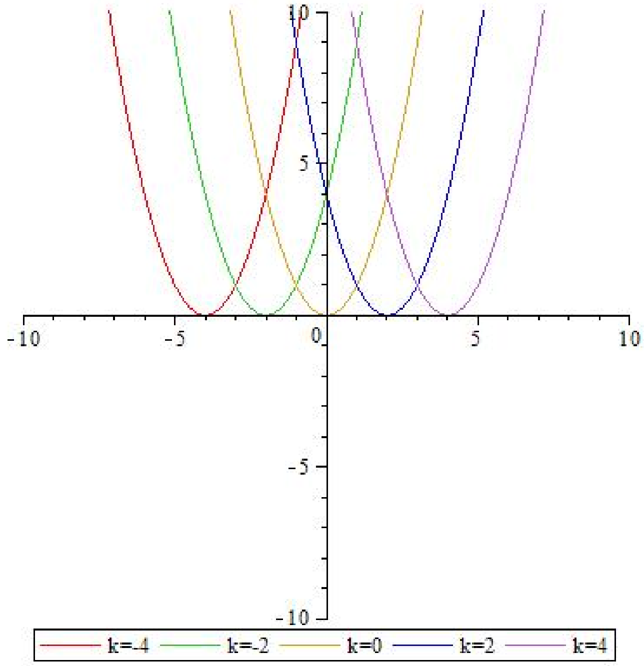
Example. Draw the graph of the functions $f_c(x) := x^2 + c$ for $c = -4, -2, 0, 2, 4$.



Example. Draw the graph of the functions $f_k(x) := -(x-3)^2 + k$ for $k = -4, -2, 0, 2, 4$.



Example. Draw the graph of the functions $f_k(x) := (x-k)^2$ for $k = -4, -2, 0, 2, 4$.



Exercise 2.4. For the below quadratic function f

1). Give the canonical form

2). If $\Delta \geq 0$ give the factorized form

3). Give the Y-intercept and X-intercepts of the graph of f

4). Give the vertex of the graph of f

5). Draw the graph of f

a) $f(x) = x^2 - 4$
b) $f(x) = x^2 - 5x + 6$
c) $f(x) = -x^2 + 6x$
d) $f(x) = -x^2 + 4x - 3$
e) $f(x) = x^2 + 4x + 8$
f) $f(x) = -x^2 - 4$
g) $f(x) = 2x^2 - 2$
h) $f(x) = 3x^2 - 6x$
i) $f(x) = -2x^2 - 4$
j) $f(x) = -3x^2 + 6x - 6$
k) $f(x) = -2x^2 - 4x - 2$
l) $f(x) = 3x^2 - 12x + 12$
m) $f(x) = x^2 - 6x + 5$
n) $f(x) = -4x^2 + 1$
o) $f(x) = x^2 - 10x + 25$
p) $f(x) = -x^2 + 16$

2.3 Polynomial functions

2.3.1 Definition of polynomial functions

Definition 2.9. A function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with $n \in \mathbb{N}$ and given $a_n, a_{n-1}, \ldots a_0 \in \mathbb{R}$ is called a polynomial function.

Remark. Linear and quadratic functions are special polynomial functions.

2.3.2 Euclidean division of polynomials in one variable

In this section we present the division algorithm for univariate polynomials with complex, real or rational coefficients. This procedure is a straightforward generalization of the long division of integers.

The steps of the polynomial division:

1). Write the dividend and the divisor polynomials in the following scheme

dividend polynomial	divisor polynomial
	quotient polynomial

2). Divide the main term of the dividend by the main term of the divisor polynomial, and write the resulting monomial to the place of the quotient

3). Multiply the above resulting monomial by the divisor, change the sign of every monomial of the result, and write the resulting polynomial below the dividend.

4). Draw a horizontal line, add the dividend to the above resulting polynomial and write the result of the addition below the line

5). Let the polynomial below the last horizontal line take the role of the dividend and repeat steps 2)-4) until the polynomial below the last horizontal line is zero or has degree strictly smaller than the degree of the divisor.

Example. Divide the polynomial $f(x) = x^4 - 3x^3 + 5x^2 + x - 5$ by $g(x) = x^2 - 2x + 2$.

$$
\begin{array}{rrrrr|l}
x^4 & -3x^3 & +5x^2 & +x & -5 & x^2 - 2x + 2 \\ \cline{6-6}
-x^4 & +2x^3 & -2x^2 & & & x^2 - x + 1 \\ \cline{1-5}
 & -x^3 & +3x^2 & +x & -5 & \\
 & x^3 & -2x^2 & +2x & & \\ \cline{1-5}
 & & x^2 & +3x & -5 & \\
 & & -x^2 & +2x & -2 & \\ \cline{1-5}
 & & & 5x & -7 &
\end{array}
$$

Remark. If in the dividend polynomial there are missing terms of lower degrees, than it is wise to include them with coefficient 0 in the scheme, so that for their like terms there is place below them.

Example. Divide the polynomial $f(x) = x^4 + 5x^2 + x - 5$ by $g(x) = x^2 - 2x + 2$.

$$
\begin{array}{rrrrr|l}
x^4 & +0x^3 & +5x^2 & +x & -5 & x^2 - 2x + 2 \\ \cline{6-6}
-x^4 & +2x^3 & -2x^2 & & & x^2 + 2x + 7 \\ \cline{1-5}
 & 2x^3 & +3x^2 & +x & -5 & \\
 & -2x^3 & +4x^2 & -4x & & \\ \cline{1-5}
 & & 7x^2 & -3x & -5 & \\
 & & -7x^2 & +14x & -14 & \\ \cline{1-5}
 & & & 11x & -19 &
\end{array}
$$

Exercise 2.5. Divide the polynomial $f(x)$ by the polynomial $g(x)$ using the procedure of Euclidean division:

a) $f(x) := x^3 + 2x^2 - 4x + 2, \quad g(x) := x^2 - x + 1$

b) $f(x) := x^5 - 3x^4 + 4x^3 + 2x^2 - 4x + 2, \quad g(x) := x^2 - x + 1$

c) $f(x) := x^5 - 3x^4 + 2x^2 - 4x + 2, \quad g(x) := x^2 - 3x + 2$

d) $f(x) := x^6 - 3x^4 + 2x^2 - 4x + 2, \quad g := x^3 - 2x + 1$

e) $f(x) := x^5 - 3x^4 + 4x^3 - 5x^2 + x + 2 \quad g(x) := x^2 - 3x + 2$

f) $f(x) := x^6 - 64, \quad g(x) := x^2 - 2x + 4$

g) $f(x) := x^5 - 2x^4 - 3x + 2, \quad g(x) := x^2 - 3x + 4$

h) $f(x) := x^5 + 2x^4 - 5x^3 + 2, \quad g := x^2 - 5x + 2$

i $f(x) := x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 2, \quad g(x) := x^2 - 4x + 3$

j) $f(x) := x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 2, \quad g(x) := x^4 - 3x^3 + x^2 - 4x + 3$

k) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 1$

l) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 1$

m) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x - 2$

n) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 2$

o) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x + 3$

p) $f(x) := x^6 + 3x^5 - 2x^4 - 5x^3 + 2x^2 - 3x + 10, \quad g(x) := x^2 - 1$

q) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 1$

r) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 1$

s) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 2$

t) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 2$

u) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x - 3$

v) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x + 3$

w) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - x - 1$

x) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 - 1$

y) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + x - 1$

z) $f(x) := x^6 - 4x^4 - x^3 + 3x - 2, \quad g(x) := x^2 + 2x - 1$

2.3.3 Computing the value of a polynomial function

To compute the value of a polynomial function $f(x)$ at c it is clearly possible by the general method, i.e. by replacing each the variable x at each occurrence of it by c, and computing the value of the resulting expression. However in this case there is a better way to compute it, by using the following Theorem:

Theorem 2.10. (Horner's scheme) *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *with* $a_n, \ldots, a_0 \in \mathbb{R}$, *and* $g(x) = x - c$ *with* $c \in \mathbb{R}$ *be two polynomials. Build the following table:*

	a_n	a_{n-1}	......	a_{i+1}	......	a_1	a_0
c	b_{n-1}	b_{n-2}	......	b_i	......	b_0	r

where

$$\begin{aligned} b_{n-1} &:= a_n \\ b_i &:= b \cdot b_{i+1} + a_{i+1} \qquad \textit{for } i = n-2, n-3, \ldots, 0 \\ r &:= c \cdot b_0 + a_0. \end{aligned} \tag{2.3}$$

Then the quotient of the polynomial division of f by g is the polynomial

$$q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0,$$

and the remainder is the constant polynomial r. Further, we also have

$$f(c) = r. \tag{2.4}$$

Remark. In other words the above theorem states that in Horner's scheme the numbers in the first line are the coefficients of the polynomial f, and the numbers computed in the second line of the scheme are the following:

- the first number is the zero of the polynomial $g(x) = x - c$
- the next numbers (except for the last one) are the coefficients of the quotient polynomial of the polynomial division of f by g
- the last number is the remainder of the above division, but it is also the value of the polynomial $f(x)$ at $x = c$.

Example. Now we compute the value of the polynomial function $f(x) = x^4 + 2x^3 + 5x^2 + x - 5$ in $x = 2$ using Horner's scheme. The first place in the first line is empty, then we list the coefficients of f. The first element in the second line of the Horner's scheme will be 2. Then we compute the consecutive elements of the second line using (2.3) to get

	1	2	5	1	-5
2	1	4	13	27	49

This means that $f(2) = 49$.

Remark. If in the dividend polynomial there are missing terms of lower degrees, than it is compulsory to include the coefficients of these terms (i.e. 0-s) in the upper row of the Horner's scheme.

Example. We compute the value of the polynomial function $f(x) = x^4 - 5x^2 + x - 5$ at $x = -2$ using Horner's scheme. The first place in the first row is empty, then we list the coefficients of f, including the coefficient 0 of x^3. The first element in the second row of the Horner's scheme will be -2. Then we compute the consecutive elements of the second line using (2.3) to get

	1	0	-5	1	-5
-2	1	-2	-1	3	-11

This means that $f(-2) = -11$.

Exercise 2.6. Compute the value of the following polynomial function $f(x)$ at the given value of the indeterminate x:

a) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 1$

b) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 2$

c) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 3$

d) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -4$

e) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -3$

f) $f(x) = x^8 + 2x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -2$

g) $f(x) = x^8 + 3x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = -1$

h) $f(x) = x^8 + 3x^7 - 13x^6 - 12x^5 + 11x^4 + 23x^3 + 59x^2 + 36x + 36, \quad x = 0$

i) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 1$

j) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -1$

k) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 2$

l) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -2$

m) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 3$

n) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = -3$

o) $f(x) = x^6 - 13x^4 - 9x^3 + 40x^2 + 81x - 36, \quad x = 4$

p) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = 1$

q) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = 2$

r) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = 3$

s) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = -1$

t) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = -2$

u) $f(x) = x^5 - 6x^3 - 6x^2 - 7x - 6, \quad x = -3$

v) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = 1$

w) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = 2$

x) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = 3$

y) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = -1$

z) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = -2$

ω) $f(x) = x^5 - 2x^2 - 9x - 6, \quad x = -3$

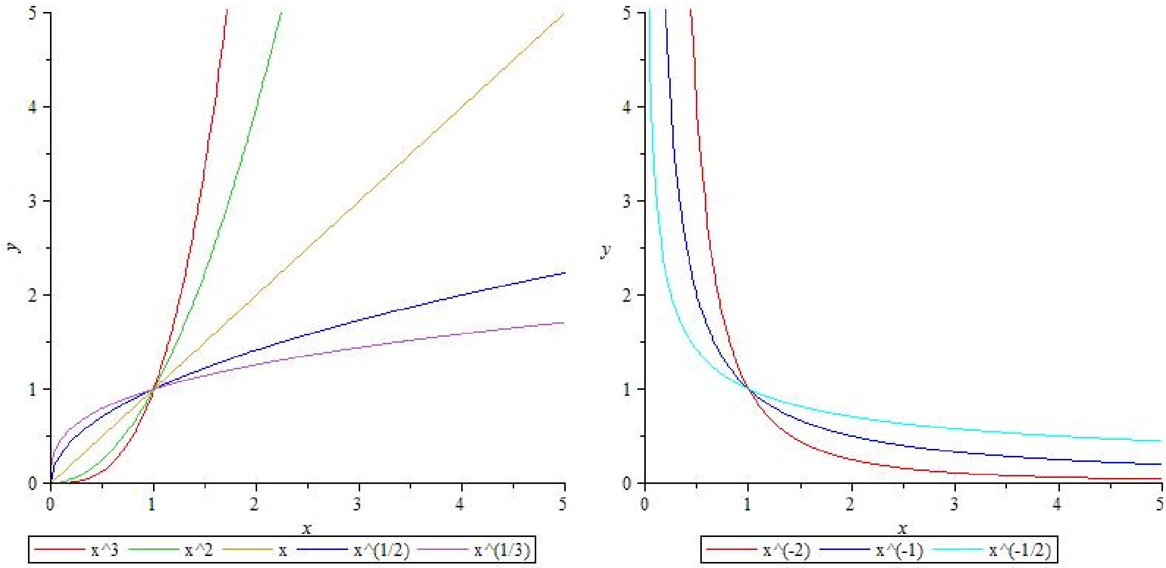
2.4 Power functions

Definition 2.11. A function $f:]0,\infty[\to \mathbb{R}$ given by

$$f(x) := Ax^r$$

with given $A \in \mathbb{R}$ and $r \in \mathbb{R}$ is called a power function.

Example. Draw the graph of the functions $f:]0,\infty[\to \mathbb{R}$, $f(x) = x^r$ for $r = 3, 2, 1, \frac{1}{2}, \frac{1}{3}$ and for $r = -2, -1, -\frac{1}{2}$.



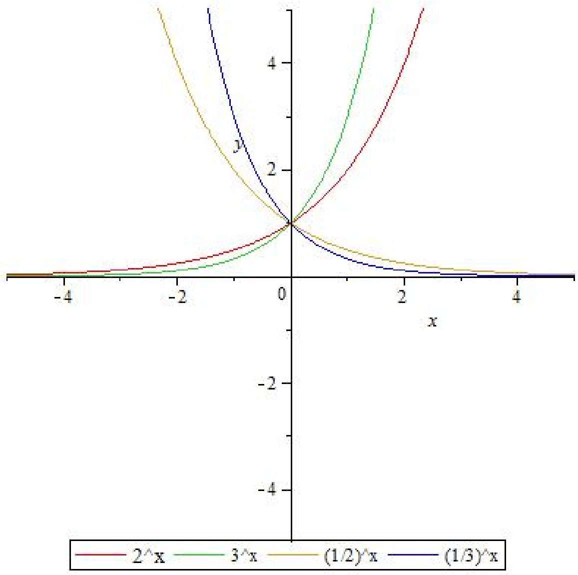
2.5 Exponential functions

Definition 2.12. A function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := Aa^x$$

with a given $a, A \in]0, \infty[$, $a \neq 1$ is called an exponential function.

Example. Draw the graph of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ for $a = 2, 3, \frac{1}{2}, \frac{1}{3}$ and for.



Definition 2.13. In the sequel we denote by e the Euler constant

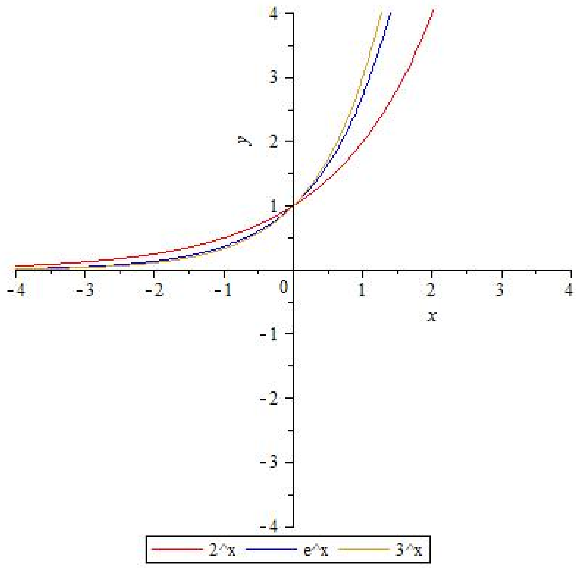
$$e := 2.71828182845904523536028747135...$$

Definition 2.14. The function $f : \mathbb{R} \to \mathbb{R}$ given by

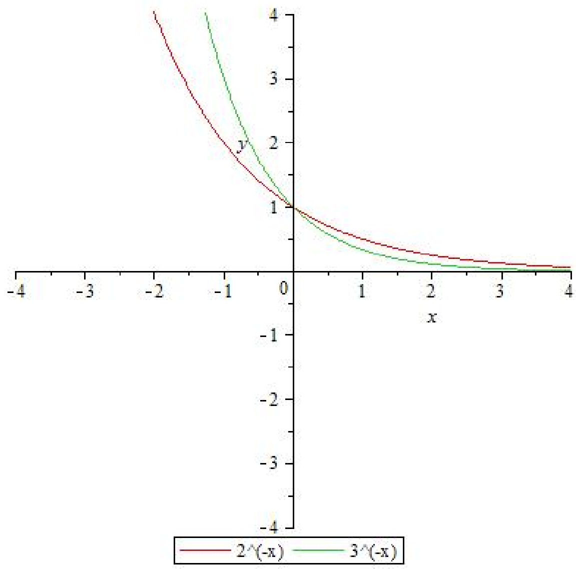
$$f(x) := e^x$$

is called the natural exponential function.

Example. Draw the graph of the functions $f(x) = e^x, g(x) = 2^x, h(x) = 3^x$.



Example. Draw the graph of the functions $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}, g(x) = f(x) = \left(\frac{1}{3}\right)^x = 3^{-x}$.



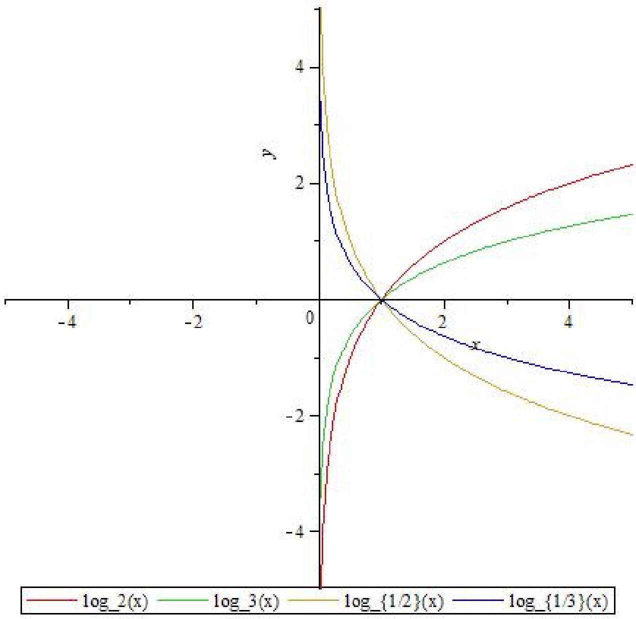
2.6 Logarithmic functions

Definition 2.15. A function $f :]0, \infty[\to \mathbb{R}$ given by

$$f(x) := \log_b x$$

with a given $b \in]0, \infty[$, $b \neq 1$ is called a logarithmic function.

Example. Draw the graph of the functions $f :]0, \infty[\to \mathbb{R}$, $f(x) = \log_b x$ for $b = 2, 3, \frac{1}{2}, \frac{1}{3}$ and for.



Definition 2.16. Recall that e denotes the Euler constant

$$e := 2.71828182845904523536028747135...$$

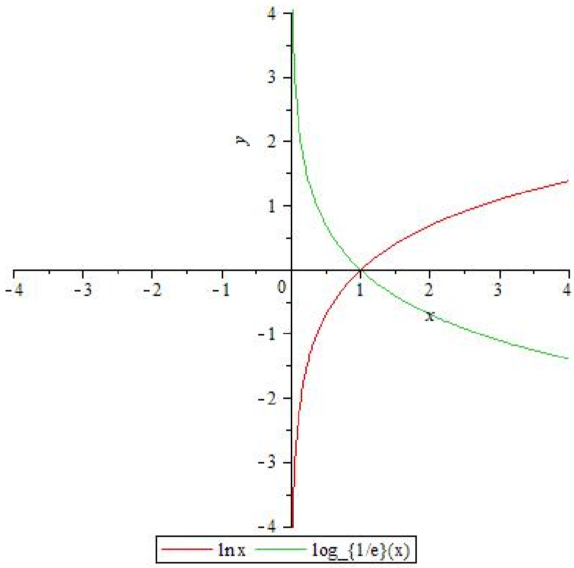
Definition 2.17. The function $f :]0, \infty[\to \mathbb{R}$ given by

$$f(x) := \log_e x$$

where e is the Euler-constant is called the natural logarithmic function. We denote the logarithm on base e by ln, and similarly the natural logarithmic function by $\ln x$.

Remark. The natural logarithmic function is sometimes also denoted by $\log x$.

Example. Draw the graph of the functions $f\ :]0,\infty[\to\mathbb{R},\ f(x) = \ln x = \log_e x$ and $g\ :]0,\infty[\to\mathbb{R},\ g(x) = \log_{1/e} x$.



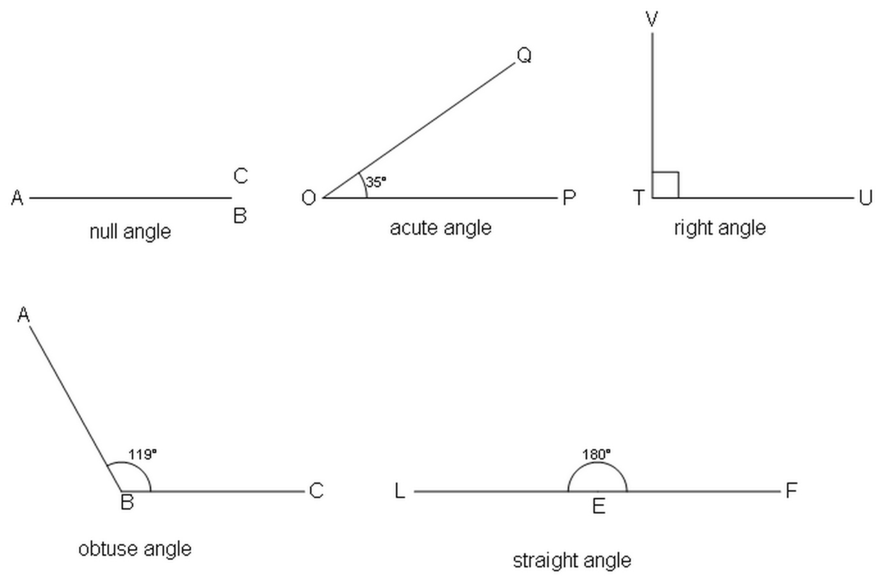
Remark. We also recall that the logarithmic function on base 10 is denoted by $\lg x := \log_{10} x$.

Chapter 3

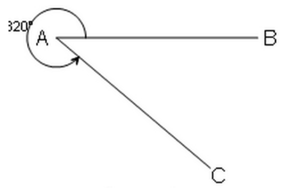
Trigonometric functions

3.1 Measures for angles: Degrees and Radians

In geometry an angle is considered to be a figure which consists of two half-lines called the sides, sharing a common endpoint, called the vertex. So basically we have the following kinds of angles:



By considering also the points of the plane between the two sides as parts of the angle, we may extend the notion of angle to reflex angles (those larger than a straight angle) and full angles:



reflex angle

full angle

Definition 3.1. Divide the full angle into 360 equal angles. The measure for one such part is defined to be one degree, and the measure of any angle will be as many degrees as many parts of 1 degree fit between the sides of the angle. The notation for degree is a circle "in the exponent" e.g. $30°$.

What is the reason for choosing the number 360? In one hand, 360 is divisible by many integers. So the half, the third, the quoter, the one fifth, on sixth, one eighth, on tenth, one twelfth of a whole circle has an integer measure, and this is very convenient. On the other hand 360 is close to the number of days in the astronomical years, which was convenient in astronomy. In geometry this measure for angles is also convenient, however, in trigonometry and analysis it is inconvenient, and there we use radians to measure the angles:

Definition 3.2. The measure of an angle in radians is the length of the arc corresponding to the angle (as a central angle) of a unit circle. This means that the measure of a full angle is 2π. When measuring angles in radians the size of the angle is a pure number without explicitly expressed unit.

Remark. In many cases the word "angle" will be used not only for the geometric figure, but also for its measure.

Theorem 3.3. *Let us have an angle of measure $x°$. Then its measure in radians is $y = x \cdot \frac{2\pi}{360}$.*

Let us have an angle of measure y radians. Then its measure in degrees is $x = y \cdot \frac{360°}{2\pi}$.

Example. The measure of the most important angles in degrees and radians:

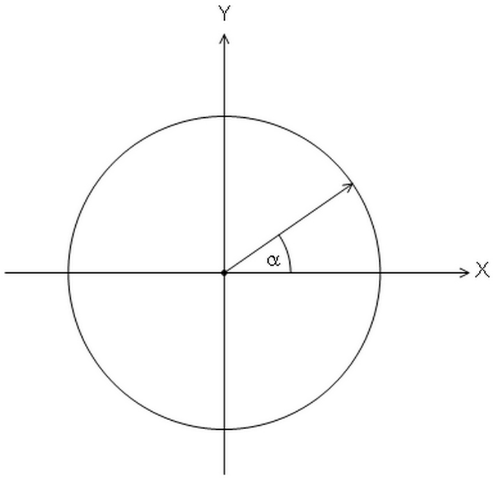
deg	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

3.2 Rotation angles

To extend the notion of angles such that every real number corresponds to the measure of an angle we do the following:

- We consider a Cartesian coordinate system in the plane and a unit-circle with the center in the origin.
- Consider the angle determined by the X-axis and a radius of this circle. The radius starts with the position in which its endpoint is the point $(0, 1)$. This angle represents the null-angle.
- Positive angles are obtained by counterclockwise rotation of this radius and negative angles are obtained by clockwise rotation of the radius. We shall speak about angles and rotations interchangeably.
- The X-axis will be called the initial side of the angle and the rotating radius the terminal side.
- The measure of a rotation may be expressed both in degrees and in radians. For example, the measure of a full revolution is $360°$ or 2π radians.
- If the rotation is less than one revolution then the measure of the rotation is just the measure of the angle determined by the *initial side* and the *terminal side*.
- If the rotation has gone through at least one revolution then the measure of the rotation is more than $360°$ or 2π radians, respectively. More precisely, if the rotation has gone through k complete revolutions and some more, than the measure of the rotation is $k \cdot 360°$ (or $2k\pi$ radians , respectively) plus the the measure of the angle determined by the initial side and the terminal side.
- If the rotation is in clockwise direction, then the measure of the rotation is negative, and it is computed analogously.

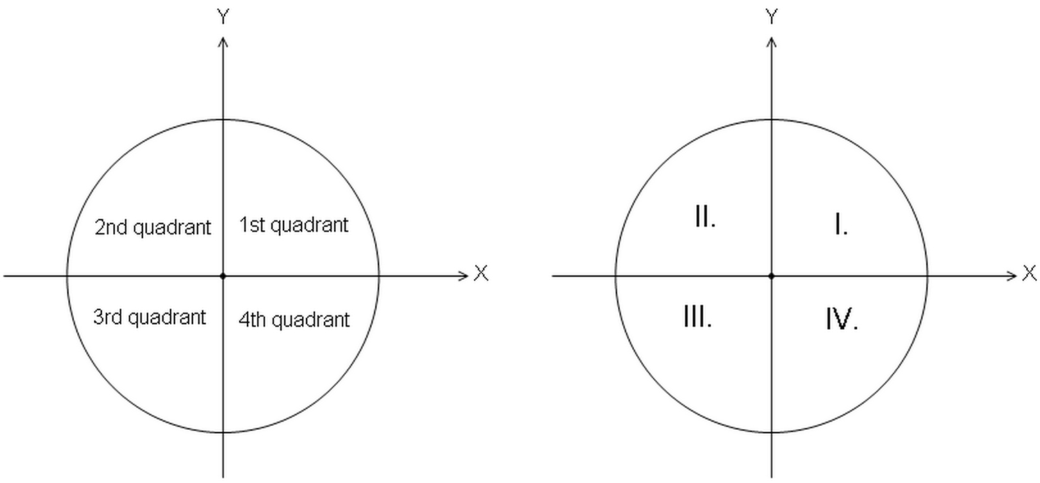
Remark. If α is measured in degrees on the picture below, then the radius which corresponds to the angle α is the same which corresponds to $k \cdot 360° + \alpha$ for any $k \in \mathbb{Z}$.



Definition 3.4. Angles which have the same terminal side are called **coterminal**. Further, we say that an **angle is on the first circle** if its measure in degrees is on the interval $[0°, 360°[$ (or alternatively, if its measure in radians is in the interval $[0, 2\pi[$).

Theorem 3.5. *Every rotation angle is coterminal with an angle on the first circle.*

The two coordinate axes divide the plane into four quadrants, as it is shown on the figure below.



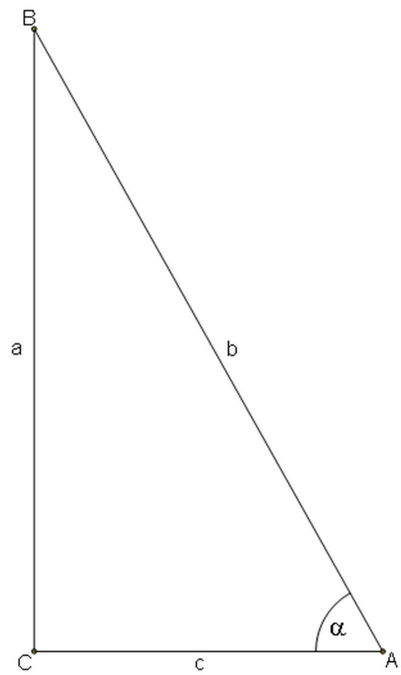
We say that an angle belongs to one of the quadrants, if its terminal side is in that quadrant.

Example. Compute the angle corresponding to the

3.3 Definition of trigonometric functions for acute angles of right triangles

3.3.1 The basic definitions

Let ABC_Δ be a right triangle, the right angle being at the vertex C. Suppose that the size of the angle at vertex A is α. Then we define the following:



The sine of α is defined by $\sin\alpha = \frac{a}{c}$

The cosine of α is defined by $\cos\alpha = \frac{b}{c}$

The tangent of α is defined by $\tan\alpha = \frac{a}{b}$

The cotangent of α is defined by $\cot\alpha = \frac{b}{a}$

The above definitions may be also formulated in the following way

Definition 3.6. Let α be one acute angle of a right triangle.

- The sine of α is defined by

$$\sin\alpha = \frac{\text{length of the leg opposite to } \alpha}{\text{length of the hypotenuse}}$$

- The cosine of α is defined by

$$\cos\alpha = \frac{\text{length of the leg adjacent to } \alpha}{\text{length of the hypotenuse}}$$

- The tangent of α is defined by

$$\tan\alpha = \frac{\text{length of the leg opposite to } \alpha}{\text{length of the leg adjacent to } \alpha}$$

- The cotangent of α is defined by

$$\cot\alpha = \frac{\text{length of the leg adjacent to } \alpha}{\text{length of the leg opposite to } \alpha}$$

Remark. The definition of trigonometric functions above are independent of the choice of the triangle. Indeed, if we have two right triangles having an acute angle of measure α, then these triangles are similar to each other. Thus the quotient of the length of the two corresponding sides in the two triangles is the same.

Example. Let the lengths of the sides of a right triangle be $3, 4, 5$. Let α be the angle opposite to the leg of length 3. Compute $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$.
Solution.

$$\sin\alpha = \frac{3}{5} = 0.6$$
$$\cos\alpha = \frac{4}{5} = 0.8$$
$$\tan\alpha = \frac{3}{4} = 0.75$$
$$\cot\alpha = \frac{4}{3} \approx 1.33$$

3.3.2 Basic formulas for the trigonometric functions defined for acute angles

Theorem 3.7. (The cofunction identities) *Let α be an acute angle in a right triangle. Then we have:*

1). $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$;
2). $\cos\alpha = \sin\left(\frac{\pi}{2} - \alpha\right)$;
3). $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$;
4). $\cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$.

Theorem 3.8. (The quotient identities) *Let α be an acute angle in a right triangle. Then we have:*

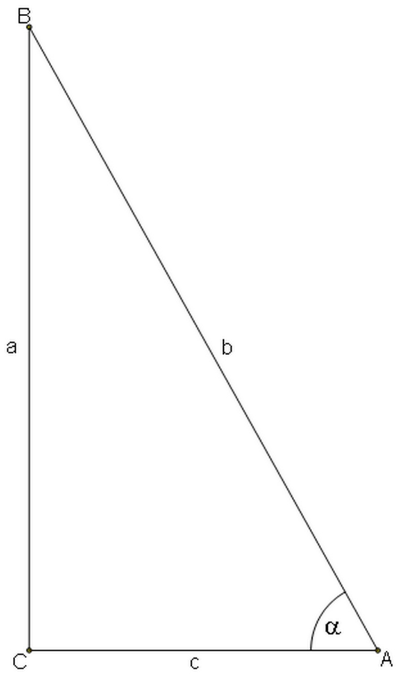
$$1).\ \tan\alpha = \frac{\sin\alpha}{\cos\alpha};$$
$$2).\ \cot\alpha = \frac{\cos\alpha}{\sin\alpha};$$
$$3).\ \cot\alpha = \frac{1}{\tan\alpha}.$$

Theorem 3.9. (The trigonometric theorem of Pythagoras in right triangles) *Let α be an acute angle in a right triangle. Then we have:*

$$\sin^2\alpha + \cos^2\alpha = 1.$$

Proof of Theorems 3.7, 3.8, 3.9. Let ABC_Δ be a right triangle, the right angle being at the vertex C and with the size of the angle at vertex A being α.

Then the size of the angle at vertex B will be $\frac{\pi}{2} - \alpha$. Using Definition 3.6 we get the cofunction identities



$$\sin\left(\frac{\pi}{2} - \alpha\right) = \frac{b}{c} = \cos\alpha$$
$$\cos\left(\frac{\pi}{2} - \alpha\right) = \frac{a}{c} = \sin\alpha$$
$$\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{b}{a} = \cot\alpha$$
$$\cot\left(\frac{\pi}{2} - \alpha\right) = \frac{a}{b} = \tan\alpha$$

Now we prove the quotient identities:

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c}\cdot\frac{c}{b} = \frac{a}{b} = \tan\alpha$$
$$\frac{\cos\alpha}{\sin\alpha} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{c}{b}\cdot\frac{a}{c} = \frac{b}{a} = \cot\alpha$$
$$\frac{1}{\tan\alpha} = \frac{1}{\frac{a}{b}} = \frac{b}{a} = \cot\alpha.$$

Finally, we prove Theorem 3.9. The Theorem of Pythagoras for the triangle ABC_Δ sates that $a^2 + b^2 = c^2$. Using this we get

$$\sin^2\alpha + \cos^2\alpha = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

which conclude the proof of Theorem 3.9. □

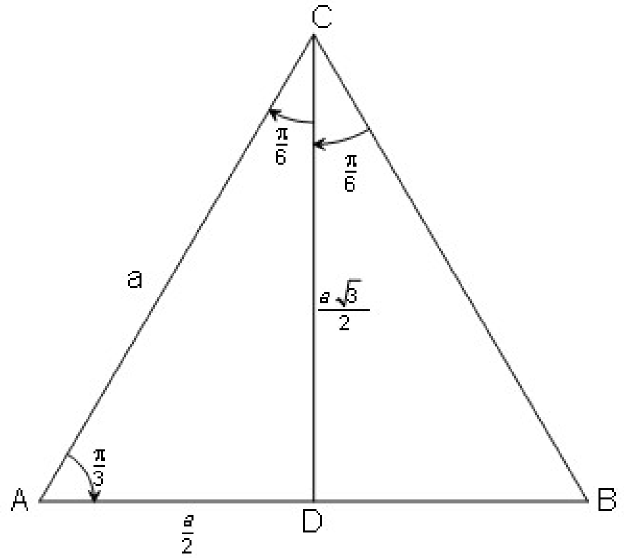
3.3.3 Trigonometric functions of special angles

In applications the angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ are of special interest. In this paragraph we compute the trigonometric functions of these angles.

Theorem 3.10. *The values of the trigonometric functions of* $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ *are summarized in the following table.*

α	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\alpha$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\alpha$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\cot\alpha$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

Proof. To prove the statements of the theorem concerning the angles $\frac{\pi}{6}$ and $\frac{\pi}{3}$ we consider an equilateral triangle ABC_Δ, and we draw its bisector CD.



Clearly all sides of ABC_Δ are of the same length, so we may write

$$AB = BC = AC = a,$$

and since the bisector of an equilateral triangle is also its median corresponding to the opposite side, we have

$$AD = \frac{a}{2}.$$

Further, CD is also the perpendicular bisector of AB, so the triangle ADC_Δ is a right triangle, with the right angle at D. In this triangle the Theorem of Pythagoras gives:

$$CD^2 = AC^2 - AD^2 = a^2 - \left(\frac{a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4},$$

so we get

$$CD = \frac{a\sqrt{3}}{2}.$$

Clearly, have $\measuredangle CAD = \measuredangle ACB = \frac{\pi}{3}$, and since CD is the bisector of the angle $\measuredangle ACB$ we also have $\measuredangle ACD = \frac{\pi}{6}$. Thus, in the right triangle ADC_Δ we have

$$\sin\frac{\pi}{3} = \frac{CD}{AC} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{6} = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$
$$\cos\frac{\pi}{3} = \frac{AD}{AC} = \frac{\frac{a}{2}}{a} = \frac{1}{2} \qquad \sin\frac{\pi}{6} = \frac{AD}{AC} = \frac{1}{2},$$

and similarly,

$$\tan\frac{\pi}{3} = \frac{CD}{AD} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3} \qquad \cot\frac{\pi}{6} = \frac{CD}{AD} = \sqrt{3}$$
$$\cot\frac{\pi}{3} = \frac{AD}{CD} = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan\frac{\pi}{6} = \frac{AD}{CD} = \frac{\sqrt{3}}{3}.$$

To prove the statements of the theorem concerning the angle $\frac{\pi}{4}$ we consider a right isosceles triangle ABC_Δ, with its right angle being at the vertex A. Since the two legs have the same length we may write

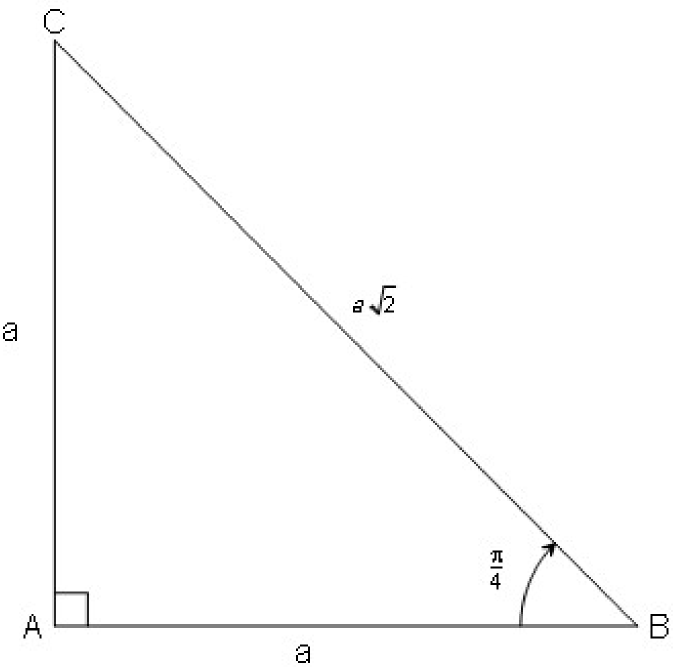
$$AB = AC = a,$$

and the Theorem of Pythagoras gives

$$BC^2 = AB^2 + AC^2 = a^2 + a^2 = 2a^2,$$

i.e. we have

$$BC = a\sqrt{2}.$$



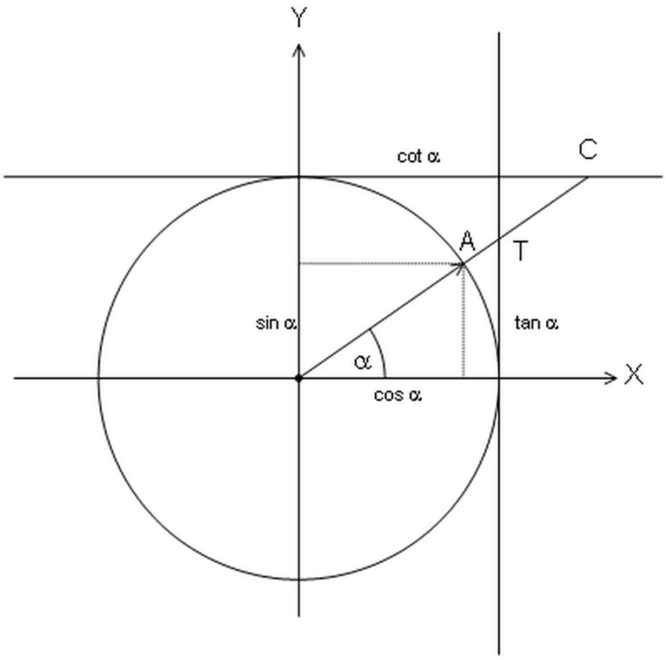
Clearly, we also have $\measuredangle ABC = \measuredangle ACB = \frac{\pi}{4}$, so from the triangle ABC_Δ using its angle $\measuredangle ABC$ we get:

$$\sin\frac{\pi}{4} = \frac{AC}{BC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = \frac{AB}{BC} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan\frac{\pi}{4} = \frac{AC}{AB} = \frac{a}{a} = a \qquad \cot\frac{\pi}{4} = \frac{AB}{BC} = \frac{a}{a} = 1.$$

This concludes the proof of our theorem. □

3.4 Definition of the trigonometric functions over $\mathbb{R}$



Definition 3.11. Consider a unit circle with a rotating radius. Let α be a rotation angle on this circle. Let the intersection point of the line of the terminal side of α by the tangent line drawn to the circle in the point $(1, 0)$, if it exists at all, be denoted by T. Similarly, let the intersection point of the line of the terminal side of α by the tangent line drawn to the circle in the point $(0, 1)$, if it exists at all, be denoted by C. We remind that the point T does not exist if $\alpha = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$, and C does not exist if $\alpha = k\pi$ for $k \in \mathbb{Z}$. Then for any $\alpha \in \mathbb{R}$ we define the trigonometric functions of α as follows:

- $\sin \alpha$ is the Y-coordinate of the endpoint A of the terminal side of α
- $\cos \alpha$ is the X-coordinate of the endpoint A of the terminal side of α
- $\tan \alpha$ is the Y-coordinate of the point T, whenever $\alpha \neq \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$
- $\cot \alpha$ is the X-coordinate of the point C, whenever $\alpha \neq k\pi$ for $k \in \mathbb{Z}$

Remark. For acute angles Definition **??** coincides with the definition of trigonometric functions in right triangles. Thus Definition **??** extends the former concept of sin, cos, tan and cot for any real number. More precisely,

- $\sin \alpha$ and $\cos \alpha$ are defined for any real number α
- $\tan \alpha$ is not defined for $\alpha = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$, but it is defined for every other $\alpha \in \mathbb{R}$

- $\cot \alpha$ is not defined for $\alpha = k\pi$ for $k \in \mathbb{Z}$, but it is defined for every other $\alpha \in \mathbb{R}$

3.5 The period of trigonometric functions

Theorem 3.12. *The trigonometric functions sine, cosine, tangent and cotangent are periodic. Further, for the period of these functions we have:*

1). $\sin$ *has period* 2π*, i.e.* $\sin(x + 2\pi) = \sin x$
and 2π *is the smallest angle with this property holding for every* $\alpha \in \mathbb{R}$*;*

2). $\cos$ *has period* 2π*, i.e.* $\cos(x + 2\pi) = \cos x$
and 2π *is the smallest angle with this property holding for every* $\alpha \in \mathbb{R}$*;*

3). $\tan$ *has period* π*, i.e.* $\tan(x + \pi) = \tan x$
and π *is the smallest angle with this property holding for every* $\alpha \in \mathbb{R}$*;*

4). $\cot$ *has period* π*, i.e.* $\cot(x + \pi) = \cot x$
and π *is the smallest angle with this property holding for every* $\alpha \in \mathbb{R}$*.*

Proof. If we investigate the figure in Definition **??** we see that for any angle α the terminal side of α and $\alpha + 2\pi$ is the same, so for α and $\alpha + 2\pi$ the point A will be the same, which proves that α and $\alpha + 2\pi$. Further, it is clear that 2π is the smallest angle with this property holding for every $\alpha \in \mathbb{R}$. This proves statements 1) and 2) of Theorem 3.12.

Next, we also see that the terminal side of α and the terminal side of $\alpha + \pi$ are on the same line, thus fixing the same points T and C, which proves $\tan(x + \pi) = \tan x$ and $\cot(x+\pi) = \cot x$. Further, it is clear again, that π is the smallest angle with this property holding for every $\alpha \in \mathbb{R}$. This proves statements 3) and 4) of Theorem 3.12. □

Corollary 3.13. *Theorem 3.12 proves that we have*

1). $\sin(\alpha + 2k\pi) = \sin\alpha$ *for every* $k \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R}$*;*

2). $\cos(\alpha + 2k\pi) = \cos\alpha$ *for every* $k \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R}$*;*

3). $\tan(\alpha + k\pi) = \tan\alpha$ *for every* $k \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R} \setminus \{(2l+1)\pi/2 \mid l \in Z\}$*;*

4). $\cot(\alpha + k\pi) = \cot\alpha$ *for every* $k \in \mathbb{Z}$ *and* $\alpha \in \mathbb{R} \setminus \{l\pi/ \mid l \in Z\}$*.*

3.6 Symmetry properties of trigonometric functions

Theorem 3.14. *The* $\sin, \tan, \cot$ *functions are odd functions, and the* $\cos$ *function is an even function. This means that we have the following formulas:*

1). $\sin(-\alpha) = -\sin\alpha$ *for every* $\alpha \in \mathbb{R}$*;*

2). $\cos(-\alpha) = \cos\alpha$ *for every* $\alpha \in \mathbb{R}$*;*

3). $\tan(-\alpha) = -\tan\alpha$ *for every* $\alpha \in \mathbb{R} \setminus \{(2k+1)\pi/2 \mid k \in Z\}$*;*

4). $\cot(-\alpha) = -\cot\alpha$ *for every* $\alpha \in \mathbb{R} \setminus \{k\pi/ \mid k \in Z\}$*.*

3.7 The sign of trigonometric functions

The sign of the trigonometric functions $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$ is determined by the situation of the terminal side of α. Whenever α belongs to one of the four quadrants the sign of the above functions is already determined, and the same is true for the cases when the terminal side of α is at border of two quadrants.

We may summarize the information about the sign of the trigonometric functions $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$ as follows:

1). **If α is in the 1^{st} quadrant** then

- $\sin\alpha > 0$
- $\cos\alpha > 0$
- $\tan\alpha > 0$
- $\cot\alpha > 0$

2). **If α is at the border of the 1^{st} and 2^{nd} quadrant**, i.e. $\alpha \in \left\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}$ then

- $\sin\alpha = 1$
- $\cos\alpha = 0$
- $\tan\alpha$ is not defined
- $\cot\alpha = 0$

3). **If α is in the 2^{nd} quadrant** then

- $\sin\alpha > 0$
- $\cos\alpha < 0$
- $\tan\alpha < 0$
- $\cot\alpha < 0$

4). **If α is at the border of the 2^{nd} and 3^{rd} quadrant**, i.e. $\alpha \in \{(2k+1)\pi \mid k \in \mathbb{Z}\}$ then

- $\sin\alpha = 0$
- $\cos\alpha = -1$
- $\tan\alpha = 0$
- $\cot\alpha$ is not defined

5). **If α is in the 3^{rd} quadrant** then

- $\sin\alpha < 0$
- $\cos\alpha < 0$
- $\tan\alpha > 0$
- $\cot\alpha > 0$

6). **If α is at the border of the 3^{rd} and 4^{th} quadrant**, i.e. $\alpha \in \left\{\frac{3\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}$ then

- $\sin\alpha = -1$
- $\cos\alpha = 0$
- $\tan\alpha$ is not defined
- $\cot\alpha = 0$

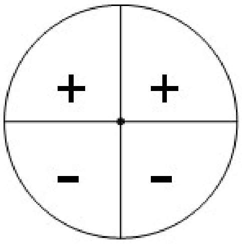
7). **If α is in the 4^{th} quadrant** then

- $\sin\alpha < 0$
- $\cos\alpha > 0$
- $\tan\alpha < 0$
- $\cot\alpha < 0$

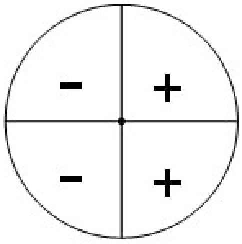
8). **If α is at the border of the 4^{th} and 1^{st} quadrant**, i.e. $\alpha \in \{2k\pi \mid k \in \mathbb{Z}\}$ then

- $\sin\alpha = 0$
- $\cos\alpha = 1$
- $\tan\alpha = 0$
- $\cot\alpha$ is not defined

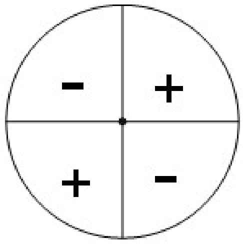
Remark. The sign of trigonometric functions can also be summarized by the following diagrams:



sin x



cos x



tan x, cot x

3.8 The reference angle

Definition 3.15. Let α be a rotation angle such that its terminal side is not on the coordinate axis. The acute angle formed by the x-axis and the terminal side of α is called the reference angle of al.

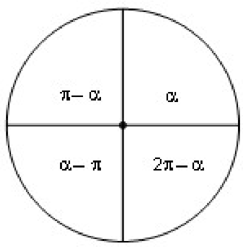
Theorem 3.16. *Let α be a rotation angle, and let β be its reference angle. Then we have the following:*

- $\sin\beta = |\sin\alpha|$
- $\cos\beta = |\cos\alpha|$
- $\tan\beta = |\tan\alpha|$
- $\cot\beta = |\cot\alpha|$

Theorem 3.17. *Let $\alpha \in [0, 2\pi[$ be a rotation angle, and let β be its reference angle. The we have the following:*

- *If α is in the first quadrant then $\beta = \alpha$;*
- *If α is in the second quadrant then $\beta = \pi - \alpha$;*
- *If α is in the third quadrant then $\beta = \alpha - \pi$;*
- *If α is in the fourth quadrant then $\beta = 2\pi - \alpha$;*

Remark. The above theorem can be summarized by the below diagram:



Reference angle of α

The theorem below is a variant of Theorem 3.16:

Theorem 3.18. *Let α be a rotation and β be its reference angle. Let* $\operatorname{sgn}(x)$ *be the sign function on* $\mathbb{R}$, *i.e.*

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then we have the following

- $\sin\alpha = \operatorname{sgn}(\sin\alpha)\sin\beta$,
- $\cos\alpha = \operatorname{sgn}(\cos\alpha)\cos\beta$,
- $\tan\alpha = \operatorname{sgn}(\tan\alpha)\tan\beta$,
- $\cot\alpha = \operatorname{sgn}(\cot\alpha)\cot\beta$.

Example. Compute the exact value of the following expressions:

1). $\sin\frac{5\pi}{4}$

2). $\cos\frac{11\pi}{6}$

3). $\tan\frac{5\pi}{3}$

Solution.

1). We shall use Theorem 3.18. First we mention that $\frac{5\pi}{4}$ is in the third quadrant, so $\operatorname{sgn}\left(\frac{5\pi}{4}\right) = -1$ and by Theorem 3.17 the reference angle of $\frac{5\pi}{4}$ is $\frac{5\pi}{4} - \pi$. So we have

$$\sin\frac{5\pi}{4} = \operatorname{sgn}\left(\sin\frac{5\pi}{4}\right)\sin\left(\frac{5\pi}{4} - \pi\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}.$$

2). Since $\frac{11\pi}{6}$ is in the fourth quadrant we have

$$\cos\frac{11\pi}{6} = \operatorname{sgn}\left(\cos\frac{11\pi}{6}\right)\cos\left(2\pi - \frac{11\pi}{6}\right) = +\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

3). Since $\frac{5\pi}{3}$ is in the fourth quadrant we have

$$\tan\frac{5\pi}{3} = \operatorname{sgn}\left(\tan\frac{5\pi}{3}\right)\tan\left(2\pi - \frac{5\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}.$$

Example. Compute the exact value of the following expressions:

1). $\sin \frac{45\pi}{6}$

2). $\cos \frac{107\pi}{4}$

3). $\tan \frac{95\pi}{3}$

Solution.

1). In contrast to the previous example the angle $\frac{45\pi}{6}$ is not on the first circle, so we cannot use Theorem **??** directly to compute the reference angle of $\frac{45\pi}{6}$. First we have to compute the angle on the first circle which corresponds to $\frac{45\pi}{6}$ (i.e. has the same terminal side as $\frac{45\pi}{6}$). This means that we have to subtract form $\frac{45\pi}{6}$ as many times 2π as it is needed the result to be on the first circle. So we have

$$\sin \frac{47\pi}{6} = \sin \left(\frac{47\pi}{6} - 3 \cdot 2\pi \right) = \sin \frac{47\pi - 6 \cdot 3 \cdot 2\pi}{6} = \sin \frac{47\pi - 36\pi}{6} = \sin \frac{11\pi}{6}$$

Now like in the previous exercise we shall use Theorem 3.18. Since $\frac{11\pi}{6}$ is in the fourth quadrant we have

$$\sin \frac{47\pi}{6} = \sin \frac{11\pi}{6} = \operatorname{sgn} \left(\sin \frac{11\pi}{6} \right) \sin \left(2\pi - \frac{11\pi}{6} \right) = -\sin \frac{\pi}{6} = -\frac{1}{2}.$$

2). Similarly to the previous example we have

$$\cos \frac{107\pi}{4} = \cos \left(\frac{107\pi}{4} - 13 \cdot 2\pi \right) = \cos \left(\frac{107\pi - 4 \cdot 13 \cdot 2\pi}{4} \right) = \cos \frac{107\pi - 104\pi}{4} =$$
$$= \frac{3\pi}{4} = \operatorname{sgn} \left(\cos \frac{3\pi}{4} \right) \cos \left(\pi - \frac{3\pi}{4} \right) = -\cos \frac{\pi}{4} = -\frac{\sqrt{2}}{2}.$$

3). Finally we use the same method to solve the last exercise:

$$\tan \frac{95\pi}{3} = \tan \left(\frac{95\pi}{3} - 15 \cdot 2\pi \right) = \tan \left(\frac{95\pi - 3 \cdot 15 \cdot 2\pi}{3} \right) = \tan \frac{95\pi - 90\pi}{3} =$$
$$= \tan \frac{5\pi}{3} = \operatorname{sgn} \left(\tan \frac{5\pi}{3} \right) \tan \left(2\pi - \frac{5\pi}{3} \right) = -\tan \frac{\pi}{3} = -\sqrt{3}.$$

We mention that instead of finding the angle on the first circle with the same terminal side as $\frac{95\pi}{3}$ we also can subtract from $\frac{95\pi}{3}$ as many times π that the result is between

0 and π. This is since the period of $\tan x$ is π.

$$\tan\frac{95\pi}{3}=\tan\left(\frac{95\pi}{3}-31\cdot\pi\right)=\tan\left(\frac{95\pi-3\cdot 31\pi}{3}\right)=\tan\frac{95\pi-93\pi}{3}=$$
$$=\tan\frac{2\pi}{3}=\operatorname{sgn}\left(\tan\frac{2\pi}{3}\right)\tan\left(\pi-\frac{2\pi}{3}\right)=-\tan\frac{\pi}{3}=-\sqrt{3}.$$

The same applies for $\cot x$, however it is important that this latter is not true in the case of $\sin x$ and $\cos x$, since their period is 2π. This means that in the case of $\sin x$ and $\cos x$ we have to subtract an integer multiple of 2π and it is not possible to replace 2π by π.

Exercise 3.1. Compute the exact value of the following expressions:

a) $\sin\frac{7\pi}{6}$ b) $\cos\frac{11\pi}{6}$ c) $\cot\frac{5\pi}{4}$

d) $\cos\frac{137\pi}{6}$ e) $\cos\frac{125\pi}{4}$ f) $\tan\frac{208\pi}{3}$

g) $\sin\frac{107\pi}{3}$ h) $\sin\frac{3110\pi}{3}$ i) $\sin\frac{1111\pi}{4}$

j) $\cos\frac{1972\pi}{3}$ k) $\cos\frac{1987\pi}{6}$ l) $\cos\frac{2012\pi}{3}$

m) $\tan\frac{1943\pi}{3}$ n) $\tan\frac{1943\pi}{6}$ o) $\cos\frac{1943\pi}{4}$

p) $\tan\frac{1975\pi}{6}$ q) $\cos\frac{1975\pi}{4}$ r) $\sin\frac{2007\pi}{4}$

s) $\sin\frac{1001\pi}{3}$ t) $\cot\frac{3724\pi}{6}$ u) $\cos\frac{2014\pi}{3}$

v) $\cos\frac{2007\pi}{2}$ w) $\sin\frac{2007\pi}{4}$ x) $\sin(2013\pi)$

y) $\cot(2007\pi)$ z) $\tan(2012\pi)$ ω) $\tan\frac{2013\pi}{2}$

3.9 Basic formulas for trigonometric functions defined for arbitrary angles

The formulas presented in Section 3.3.2 generalize automatically for trigonometric functions defined for arbitrary angles.

3.9.1 Cofunction formulas

Theorem 3.19. *Let α be a rotation angle for which the trigonometric functions in the below formulas are defined. Then we have the following formulas which connect trigonometric functions of α to cofunctions of $\frac{\pi}{2}-\alpha$:*

1). $\sin\alpha = \cos\left(\frac{\pi}{2}-\alpha\right)$;

2). $\cos\alpha = \sin\left(\frac{\pi}{2}-\alpha\right)$;

3). $\tan\alpha = \cot\left(\frac{\pi}{2}-\alpha\right)$;

4). $\cot\alpha = \tan\left(\frac{\pi}{2}-\alpha\right)$.

3.9.2 Quotient identities

Theorem 3.20. *Let α be a rotation angle for which the trigonometric functions in the below formulas are defined. Then we have:*

1). $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$;

2). $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$;

3). $\cot\alpha = \frac{1}{\tan\alpha}$.

3.9.3 The trigonometric theorem of Pythagoras

Theorem 3.21. *Let α be a rotation angle. Then we have:*

$$\sin^2\alpha + \cos^2\alpha = 1.$$

3.10 Trigonometric functions of special rotation angles

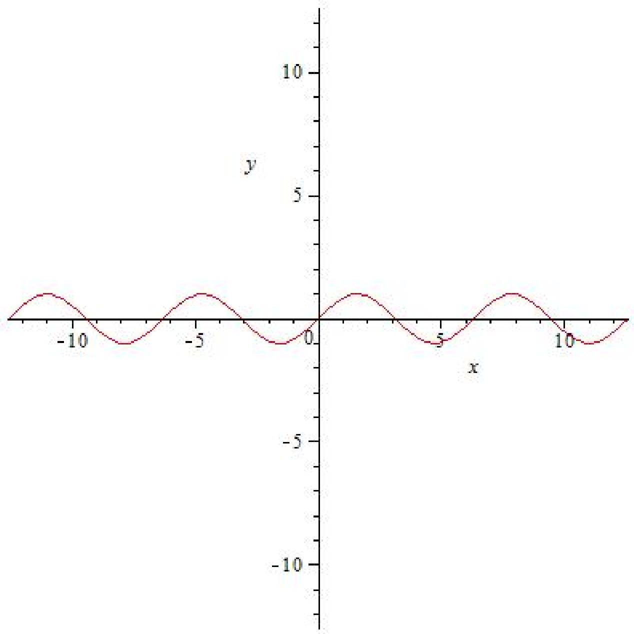
In Section **??** we computed the values of the trigonometric functions for the acute angles $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, which are of special interest for applications. In this paragraph we extend this for rotation angles with their reference angles being $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

Theorem 3.22. *The values of the trigonometric functions of some important angles are summarized in the following table.*

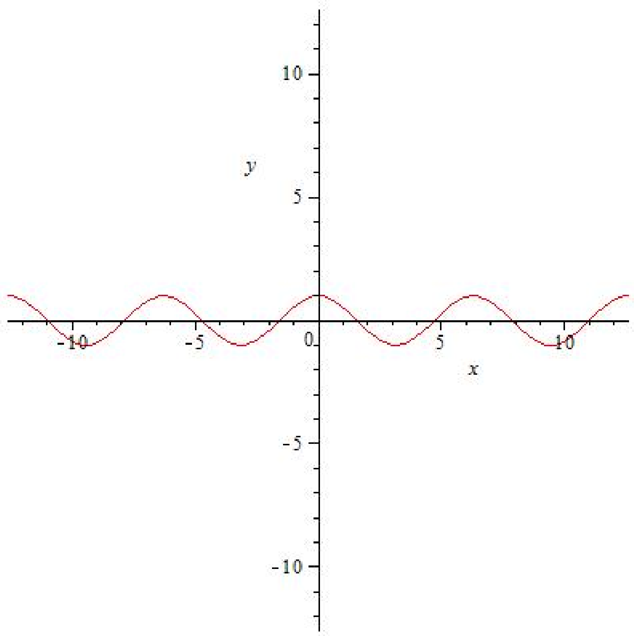
α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\vert$	0	$\vert$
$\cot\alpha$	$\vert$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\vert$	0

3.11 The graph of trigonometric functions

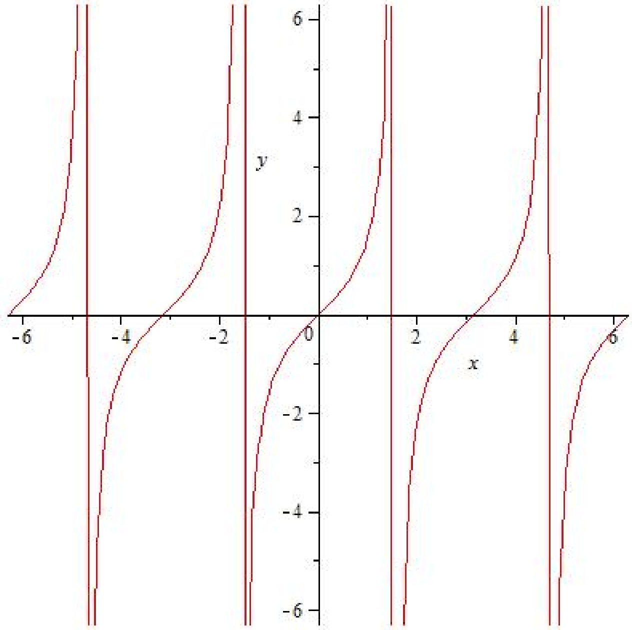
3.11.1 The graph of $\sin x$



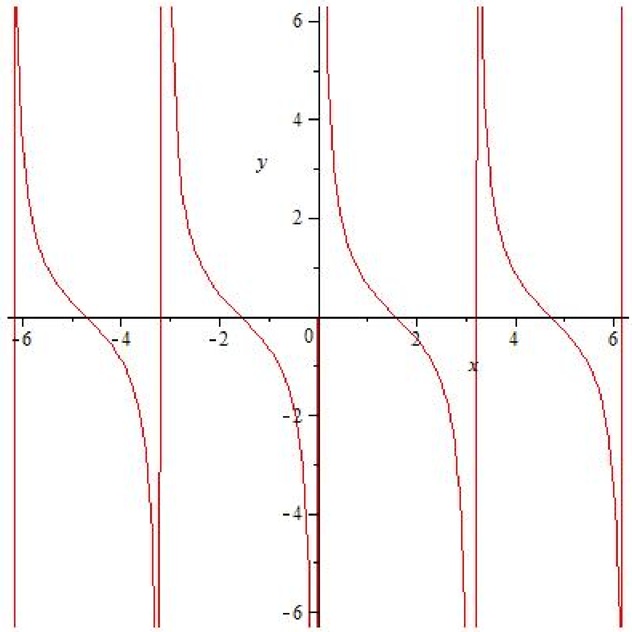
3.11.2 The graph of $\cos x$



3.11.3 The graph of $\tan x$



3.11.4 The graph of $\cot x$



3.12 Sum and difference identities

3.12.1 Sum and difference identities of the function sin and cos

Theorem 3.23. *Let α, β be two arbitrary real numbers. Then we have the following formulas:*

1). $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$;

2). $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$;

3). $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$;

4). $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Proof. We start with the proof of formula 4). Since $\cos(\alpha-\beta) = \cos(-(\alpha-\beta)) = \cos(\beta-\alpha)$ without loss of generality we may assume that $\alpha \geq \beta$. Let us denote by A the endpoint of terminal side of the angle α and by B the endpoint of terminal side of the angle β. Then the coordinates of these points are $A(\cos\alpha, \sin\alpha)$ and $B(\sin\beta, \cos\beta)$. The distance between A and B is

$$\begin{aligned} d(A,B) &= \sqrt{(\cos\beta-\cos\alpha)^2 + (\sin\alpha-\sin\beta)^2} = \\ &= \sqrt{\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta - 2\sin\beta\sin\alpha + \sin^2\alpha} = \\ &= \sqrt{(\sin^2\alpha+\cos^2\alpha) + (\sin^2\beta+\cos^2\beta) - 2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)} = \\ &= \sqrt{2-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)} \end{aligned}$$

□

Further, the size of the angle AOB is just $\alpha-\beta$, and if we rotate this angle in such a way that the side OB is moved to the x-axis (i.e. the image B' of B lies on the x-axis), then the side OA is moved to OA' such that OA' is just the terminal side of the rotation angle $\alpha-\beta$. So we have

$$\begin{aligned} d(A',B') &= \sqrt{(\cos(\alpha-\beta)-1)^2 + (\sin\alpha-\beta-0)^2} = \\ &= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} = \\ &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)) + 1 - 2\cos(\alpha-\beta)} = \\ &= \sqrt{2-2\cos(\alpha-\beta)} \end{aligned}$$

Further, by the isometric property of the rotations we have $d(A', B') = d(A, B)$, which gives

$$\sqrt{2 - 2\cos(\alpha - \beta)} = \sqrt{2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}.$$

Now rising to the square both sides, subtracting 2 from both sides and dividing them by (-2) we get the required identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Now the proof of the third identity is straightforward:

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) = \\ &= \cos\alpha\cos\beta + \sin\alpha(-\sin\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

The proof of the first identity is

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left(\frac{\pi}{2} - (\alpha + \beta)\right) = \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right) = \\ &= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta = \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta.\end{aligned}$$

Finally, the proof of the second identity is

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

Example. Compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.
Solution. We do not know the values of the exact value of the trigonometric functions of $\frac{\pi}{12}$ from tables, since $\frac{\pi}{12}$ is not an "important" angle, however since

$$\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$$

and we know the exact values of the trigonometric functions on $\frac{\pi}{4}$ and $\frac{\pi}{6}$ from our tables, we are able to compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$ using Theorem 3.23:

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3} - 1)}{4}.\end{aligned}$$

and

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4}.\end{aligned}$$

3.12.2 Sum and difference identities of the function tan and cot

Theorem 3.24. *Let α, β be two arbitrary real numbers such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$;

2). $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$;

3). $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$;

4). $\cot(\alpha-\beta) = \frac{-\cot\alpha\cot\beta-1}{\cot\alpha-\cot\beta}$.

Proof. First we prove formula 1) for $\tan(\alpha+\beta)$:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

Next let us prove 2):

$$\tan(\alpha-\beta) = \tan(\alpha+(-\beta)) = \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$
$$= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Now we present the proof of 3)

$$\cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$
$$= \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

We mention that 3) can also be proved using formula 1):

$$\cot(\alpha+\beta) = \frac{1}{\cot(\alpha+\beta)} = \frac{1}{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$
$$= \frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta} = \frac{1-\frac{1}{\cot\alpha}\frac{1}{\cot\beta}}{\frac{1}{\cot\alpha}+\frac{1}{\cot\beta}}$$
$$= \frac{\frac{\cot\alpha\cot\beta-1}{\cot\alpha\cot\beta}}{\frac{\cot\beta+\cot\alpha}{\cot\alpha\cot\beta}} = \frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$$

Finally we prove 4):

$$\cot(\alpha-\beta)=\cot(\alpha+(-\beta))=\frac{\cot\alpha\cot(-\beta)-1}{\cot\alpha+\cot(-\beta)}$$
$$=\frac{-\cot\alpha\cot\beta-1}{\cot\alpha-\cot\beta}$$

□

Example. Compute $\tan\frac{\pi}{12}$ and $\cot\frac{\pi}{12}$.
Solution. We use

$$\frac{\pi}{12}=\frac{\pi}{4}-\frac{\pi}{6}$$

and Theorem 3.24:

$$\tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}=$$
$$=\frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}=\frac{3-\sqrt{3}}{3+\sqrt{3}}.$$

Clearly, we may proceed similarly to compute $\cot\frac{\pi}{12}$, but it is much easier to use the fact that $\cot\alpha$ is the reciprocal of $\tan\alpha$ (whenever $\tan\alpha$ is non-zero):

$$\cot\frac{\pi}{12}=\frac{1}{\tan\frac{\pi}{12}}=\frac{1}{\frac{3-\sqrt{3}}{3+\sqrt{3}}}=\frac{3+\sqrt{3}}{3-\sqrt{3}}.$$

3.13 Multiple angle identities

3.13.1 Double angle identities

Theorem 3.25. *Let α be an arbitrary real number such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\sin(2\alpha) = 2\sin\alpha\cos\alpha$

2). $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

3). $\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$

4). $\cot(2\alpha) = \frac{\cot^2 -1}{2\cot\alpha}$

Proof. For proving this theorem we use the corresponding formulas from Theorem 3.23 and Theorem 3.24. First we prove formula 1):

$$\sin(2\alpha) = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 2\sin\alpha\cos\alpha$$

Now we prove 2):

$$\cos(2\alpha) = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \cos^2\alpha - \sin^2\alpha$$

For proving the other two statements of formula 2) we also need to use the trigonometric theorem of Pythagora, i.e. the formula

$$\sin^2\alpha + \cos^2 = 1.$$

So we get

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

Now we prove formulas 3) and 4)

$$\tan(2\alpha) = \tan(\alpha+\alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\cdot\tan\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\cot(2\alpha) = \cot(\alpha+\alpha) = \frac{\cot\alpha\cdot\cot\alpha - 1}{\cot\alpha + \cot\alpha} = \frac{\cot^2 -1}{2\cot\alpha}$$

□

Example. Compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

Solution. Recall that we already have computed $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$ in Section 3.12.1, using $\frac{\pi}{12}=\frac{\pi}{4}-\frac{\pi}{6}$ and Theorem 3.23, and we got the result:

$$\sin\frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{2}(\sqrt{3}-1)}{4}, \cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}=\frac{\sqrt{2}(\sqrt{3}+1)}{4}.$$

Now we wish to use identity 2) of Theorem 3.25 to compute $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$. Indeed, we have

$$\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}=\cos\left(2\cdot\frac{\pi}{12}\right)=1-2\sin^2\frac{\pi}{12}.$$

Expressing $\sin^2\frac{\pi}{12}$ from this expression we get

$$\sin^2\frac{\pi}{12}=\frac{2-\sqrt{3}}{4},$$

which together with the fact that $\frac{\pi}{12}$ is in the first quadrant so $\sin\frac{\pi}{12}$ is positive, means that

$$\sin\frac{\pi}{12}=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{\sqrt{2-\sqrt{3}}}{2}.$$

Now this result looks different of the previously obtained $\frac{\sqrt{6}-\sqrt{2}}{4}$, so it seems that one of the solutions is wrong. However, this is not the case. As the below computation sows, the two results represent exactly the same real number:

$$\frac{\sqrt{2-\sqrt{3}}}{2}=\frac{2(\sqrt{2-\sqrt{3}})}{4}=\frac{\sqrt{2}\cdot\sqrt{2(2-\sqrt{3})}}{4}=$$
$$=\frac{\sqrt{2}\cdot\sqrt{4-2\sqrt{3}}}{4}=\frac{\sqrt{2}\cdot\sqrt{3-2\sqrt{3}+1}}{4}=\frac{\sqrt{2}\cdot\sqrt{(\sqrt{3}-1)^2}}{4}=\frac{\sqrt{2}\cdot(\sqrt{3}-1)}{4}$$

Exercise 3.2. Applying sum, difference and double angle identities compute the exact values of the following expressions:

a) $\sin\frac{5\pi}{12}$	b) $\cos\frac{5\pi}{12}$	c) $\tan\frac{5\pi}{12}$
d) $\cot\frac{5\pi}{12}$	e) $\sin\frac{\pi}{8}$	f) $\cos\frac{\pi}{8}$
g) $\sin\frac{7\pi}{12}$	h) $\cos\frac{7\pi}{12}$	i) $\tan\frac{7\pi}{12}$
j) $\cot\frac{7\pi}{12}$	k) $\sin\frac{3\pi}{8}$	l) $\cos\frac{3\pi}{8}$

3.13.2 Triple angle identities

Theorem 3.26. *Let α be an arbitrary real number such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\sin(3\alpha) = 3\sin\alpha\cos^2\alpha - \sin^3 = 3\sin\alpha - 4\sin^3\alpha$

2). $\cos(3\alpha) = \cos^3\alpha - 3\sin^2\alpha\cos\alpha = 4\cos^3\alpha - 3\cos\alpha$

3). $\tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$

4). $\cot(3\alpha) = \frac{\cot^3\alpha - 3\cot\alpha}{3\cot^2\alpha - 1}$

Proof. First we prove formula 1) we use formula 1) from Theorem 3.23, formulas 1) and 2) from Theorem 3.25, and the trigonometric theorem of Pythagora:

$$\begin{aligned}\sin(3\alpha) &= \sin(2\alpha+\alpha) = \sin(2\alpha)\cos\alpha + \cos(2\alpha)\sin\alpha = \\ &= 2\sin\alpha\cos\alpha\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = \\ &= 2\sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin^3\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha = \\ &= 3\sin\alpha(1-\sin^2\alpha) - \sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha\end{aligned}$$

Similarly, to prove formula 1) we need formula 3) from Theorem 3.23, formulas 1) and 2) from Theorem 3.25, and the trigonometric theorem of Pythagora:

$$\begin{aligned}\cos(3\alpha) &= \cos(2\alpha+\alpha) = \cos(2\alpha)\cos\alpha - \sin(2\alpha)\sin\alpha = \\ &= (\cos^2\alpha - \sin^2\alpha)\cos\alpha - 2\sin\alpha\cos\alpha\sin\alpha = \\ &= \cos^3\alpha - \sin^2\alpha\cos\alpha - 2\sin^2\alpha\cos\alpha = \cos^3\alpha - 3\sin^2\alpha\cos\alpha = \\ &= \cos^3\alpha - 3(1-\cos^2\alpha)\cos\alpha = \cos^3\alpha - 3\cos\alpha + 3\cos^3\alpha = 4\cos^3\alpha - 3\cos\alpha.\end{aligned}$$

Now we prove formula 3) using formula 1) of Theorem 3.24 and formula 3) of Theorem 3.25:

$$\begin{aligned}\tan(3\alpha) = \tan(2\alpha+\alpha) &= \frac{\tan(2\alpha)+\tan\alpha}{1-\tan(2\alpha)\cdot\tan\alpha} = \\ &= \frac{\frac{2\tan\alpha}{1-\tan^2\alpha} + \tan\alpha}{1 - \frac{2\tan\alpha}{1-\tan^2\alpha}\cdot\tan\alpha} = \frac{\frac{2\tan\alpha+\tan\alpha-\tan^3\alpha}{1-\tan^2\alpha}}{\frac{1-\tan^2\alpha-2\tan^2\alpha}{1-\tan^2\alpha}} = \\ &= \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}\end{aligned}$$

Finally, we prove formula 4) by using formula 3) of Theorem 3.24 and formula 4) of Theorem 3.25:

$$\begin{aligned}\cot(3\alpha) = \cot(2\alpha+\alpha) &= \frac{\cot(2\alpha)\cdot\cot\alpha - 1}{\cot(2\alpha)+\cot\alpha} = \\ &= \frac{\frac{\cot^2\alpha-1}{2\cot\alpha}\cdot\cot\alpha - 1}{\frac{\cot^2\alpha-1}{2\cot\alpha}+\cot\alpha} = \frac{\frac{\cot^2\alpha-1}{2}\cdot -1}{\frac{\cot^2\alpha-1+2\cot^2\alpha}{2\cot\alpha}} = \\ &= \frac{\frac{\cot^2\alpha-3}{2}}{\frac{3\cot^2\alpha-1}{2\cot\alpha}} = \frac{\cot^2\alpha-3}{2}\cdot\frac{2\cot\alpha}{3\cot^2\alpha-1} = \\ &= \frac{\cot^3\alpha-3\cot\alpha}{3\cot^2\alpha-1}.\end{aligned}$$

□

3.14 Half-angle identities

Theorem 3.27. *Let α be an arbitrary real number such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$

2). $\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

3). $\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$

4). $\cot^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{1-\cos\alpha}1+\cos\alpha$

5). $\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$

6). $\cot\frac{\alpha}{2} = \frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}$

Proof. To prove the half-angle formulas we basically use identity 2) of Theorem 3.25, which applied for α instead of 2α gives

$$\cos(\alpha) = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}.$$

Expressing $\sin^2\frac{\alpha}{2}$ from the above equation we get

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

meanwhile expressing $\cos^2\frac{\alpha}{2}$ leads to the identity

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}.$$

These conclude the proof of identities 1) and 2). By dividing the two latter identities we just get 3) and 4). So it remains to prove 5) and 6). To do so first we mention that by the trigonometric theorem of Pythagoras we have $\sin^2\alpha + \cos^2\alpha = 1$, which means that $\sin^2\alpha = 1 - \cos^2\alpha$, and thus we have

$$\sin^2\alpha = (1-\cos\alpha)(1+\cos\alpha).$$

Thus, whenever $\sin\alpha \neq 0$, we get

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

and

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1-\cos\alpha}.$$

However, we mention that, by $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$, whenever $\tan\frac{\alpha}{2}$ and $\cot\frac{\alpha}{2}$ is defined we have $\sin\alpha \neq 0$. So the second equality of both 5) and 6) is proved.

To conclude the proof of this theorem we write

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{1-(1-2\sin^2\frac{\alpha}{2})}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2},$$

and

$$\frac{1+\cos\alpha}{\sin\alpha} = \frac{1+2\cos^2\frac{\alpha}{2}-1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \cot\frac{\alpha}{2}.$$

This concludes the proof of the theorem. □

3.15 Product-to-sum identities

Theorem 3.28. *Let α be an arbitrary real number such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$

2). $\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

3). $\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$

4). $\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$

5). $\tan\alpha\tan\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)}$

6). $\cot\alpha\cot\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}$

7). $\tan\alpha\cot\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)}$

8). $\cot\alpha\tan\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{\sin(\alpha+\beta)+\sin(\alpha-\beta)}$

Proof. To prove the above "product-to-sum" identities 1)-4) we start with the right-hand sides, and we transform them until we reach the formula on the left-hand side. We start with the proof of identity 1):

$$\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2} = \frac{(\cos\alpha\cos\beta+\sin\alpha\sin\beta)-(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}{2} =$$
$$= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta-\cos\alpha\cos\beta+\sin\alpha\sin\beta}{2} =$$
$$= \frac{2\sin\alpha\sin\beta}{2} = \sin\alpha\sin\beta.$$

Similarly, we get identity 2) by:

$$\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2} = \frac{(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)}{2} =$$
$$= \frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\alpha\cos\beta-\sin\alpha\sin\beta}{2} =$$
$$= \frac{2\cos\alpha\cos\beta}{2} = \cos\alpha\cos\beta.$$

The proof of identity 3) is

$$\begin{aligned}\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} &= \frac{(\sin\alpha\cos\beta+\cos\alpha\sin\beta)+(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{2} = \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta}{2} = \\ &= \frac{2\sin\alpha\cos\beta}{2} = \sin\alpha\cos\beta,\end{aligned}$$

and for the proof of identity 4) we have

$$\begin{aligned}\frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2} &= \frac{(\sin\alpha\cos\beta+\cos\alpha\sin\beta)-(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{2} = \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta}{2} = \\ &= \frac{2\cos\alpha\sin\beta}{2} = \cos\alpha\sin\beta.\end{aligned}$$

For the proof of identities 5)-8) we have to divide correspondingly two of the identities 1)-4). So we have

$$\begin{aligned}\tan\alpha\tan\beta &= \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta} = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)}\\ \cot\alpha\cot\beta &= \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}\\ \tan\alpha\cot\beta &= \frac{\sin\alpha\cos\beta}{\cos\alpha\sin\beta} = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)}\\ \cot\alpha\tan\beta &= \frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta} = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{\sin(\alpha+\beta)+\sin(\alpha-\beta)}\end{aligned}$$

□

3.16 Sum-to-product identities

Theorem 3.29. *Let α be an arbitrary real number such that the functions in the below formulas are defined for their arguments. Then we have:*

1). $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

2). $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

3). $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

4). $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Proof. Again we shall start from the right-hand side and we will do equivalent transformations of the expressions until we reach the left-hand side of the corresponding formula. We start with identity 1):

$$
\begin{aligned}
&2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right) = \\
&\quad = 2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\cdot\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \\
&\quad = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}+2\sin^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}+2\cos^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin^2\frac{\beta}{2} \\
&\quad = \sin\alpha\cos^2\frac{\beta}{2}+\sin\beta\sin^2\frac{\alpha}{2}+\sin\beta\cos^2\frac{\alpha}{2}+\sin\alpha\sin^2\frac{\beta}{2} = \\
&\quad = \sin\alpha\left(\sin^2\frac{\beta}{2}+\cos^2\frac{\beta}{2}\right)+\sin\beta\left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\right) = \sin\alpha+\sin\beta.
\end{aligned}
$$

Similarly, we prove identity 2):

$$
\begin{aligned}
&2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} = 2\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \\
&\quad = 2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\cdot\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \\
&\quad = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}-2\sin^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}-2\cos^2\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\beta}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin^2\frac{\beta}{2} \\
&\quad = \sin\alpha\cos^2\frac{\beta}{2}-\sin\beta\sin^2\frac{\alpha}{2}-\sin\beta\cos^2\frac{\alpha}{2}+\sin\alpha\sin^2\frac{\beta}{2} = \\
&\quad = \sin\alpha\left(\sin^2\frac{\beta}{2}+\cos^2\frac{\beta}{2}\right)-\sin\beta\left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\right) = \sin\alpha-\sin\beta.
\end{aligned}
$$

The proof of identity 3) is the following:

$$
\begin{aligned}
2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} &= 2\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right) = \\
&= 2\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\cdot\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \\
&= 2\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}-2\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}\left(1-\sin^2\frac{\beta}{2}\right)-2\left(1-\cos^2\frac{\alpha}{2}\right)\sin^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-2\sin^2\frac{\beta}{2}+2\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}-1+1-2\sin^2\frac{\beta}{2} = \cos\alpha+\cos\beta.
\end{aligned}
$$

Finally, the proof of identity 4) is:

$$
\begin{aligned}
-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} &= -2\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right) = \\
&= -2\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\cdot\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = \\
&= 2\cos^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}-2\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}\left(1-\cos^2\frac{\beta}{2}\right)-2\left(1-\cos^2\frac{\alpha}{2}\right)\cos^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}-2\cos^2\frac{\beta}{2}+2\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2} = \\
&= 2\cos^2\frac{\alpha}{2}-1+1-2\cos^2\frac{\beta}{2} = \cos\alpha-\cos\beta.
\end{aligned}
$$

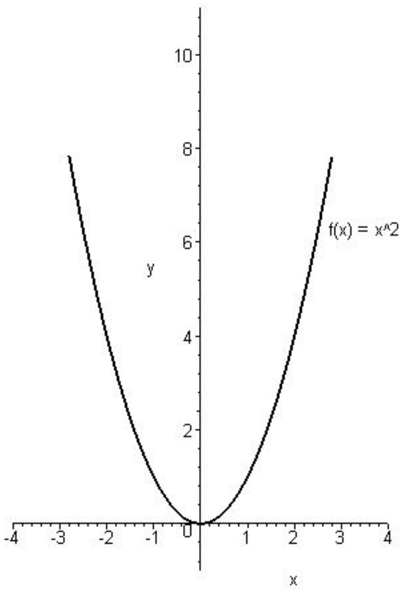
□

Chapter 4

Transformations of functions

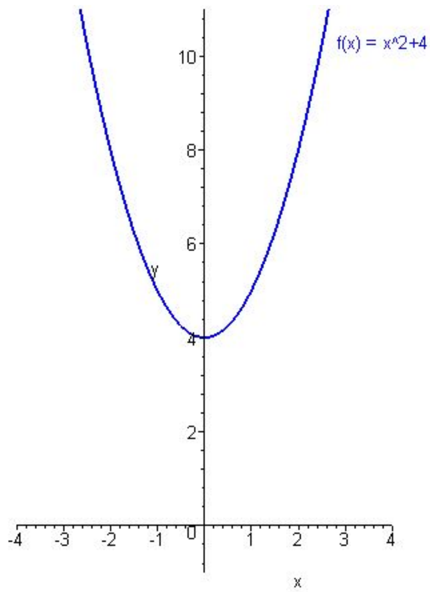
4.1 Shifting graphs of functions

In this section, we will investigate how we can obtain the graph of the functions $f(x) + c$ and $f(x + c)$ from the graph of f, where c is a real constant.
Since we already have some experiences with drawing graphs of functions, let us consider the graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.



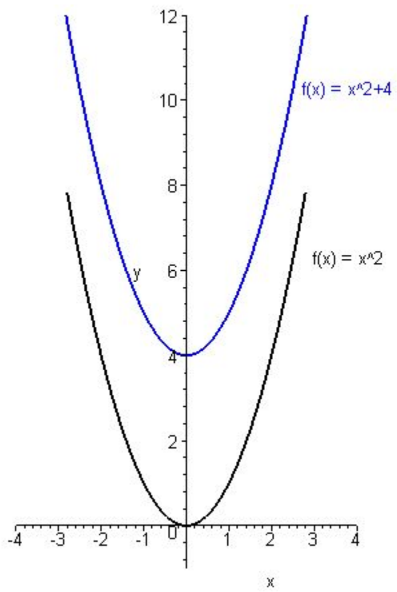
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

Let us modify the function above and sketch the graph of $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 + 4$ now.



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 + 4$.

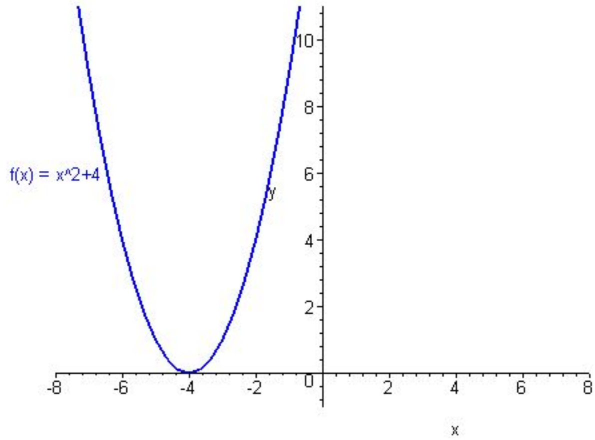
Drawing both graphs above in the same coordinate system, we obtain the following:



Graph of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 + 2$.

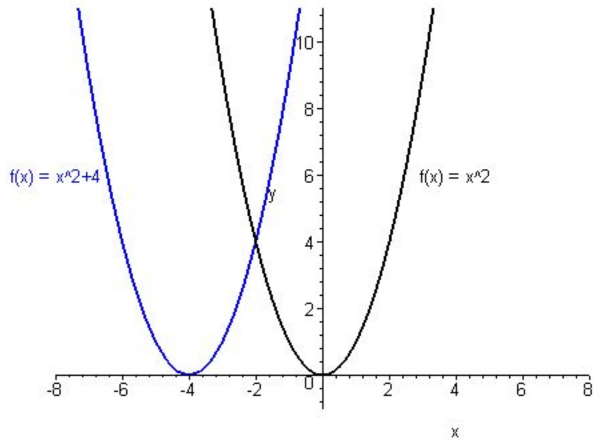
It is easy to see, that, starting with a point (x, y) of the graph of the function f, we can obtain the element (x, z) of the graph of g by $z = y + 2$. Since this property is valid for all pairs of points (x, y) and (x, z) of the graphs of our functions, we may obtain the graph of g if we shift (or move, or translate) the graph of f upwards by 4 units. Obviously,

this method works for all positive integers c. It is also clear, that, in the case when c is a negative real number, we have to shift (or move, or translate) the graph of f downwards by $|c|$ units (where $|c|$ denotes the absolute value of c). Finally, it is trivial, that, if $c = 0$ then the functions f and g coincide.
Let us sketch now the graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = (x + 4)^2$.



Graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = (x + 4)^2$.

We shall illustrate the graphs of f and h in the same coordinate system now.



Graph of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = (x + 4)^2$.

Similarly to the case above, we can see, that we may obtain the graph of h if we shift (or move, or translate) the graph of f to the left by 4 units. This method also works for each positive integer c. Furthermore, if c is a negative real number, we have to shift (or move, or translate) the graph of f to the right by $|c|$ units. (Here, it is also true that, in the case when $c = 0$ then the functions f and g are equal.)
It is easy to see that the argumentation above is independent from the concrete functions f, g and h, it only depends on the connection between the functions. Thus, we can give a general method for the types of function transformations considered above. In order describe this method, let us denote by D_c the translation of the set D by c units. More precisely, if $D \subseteq \mathbb{R}$ is a set and c is a real number, $D_c \subseteq \mathbb{R}$ denotes the set $D_c = \{s + c \mid s \in D\}$.

Let c be a real number, $D \subseteq \mathbb{R}$ be a set and let us consider a function $f : D \to \mathbb{R}$.

– In the case when c is nonnegative, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = f(x)+c$ by shifting (or moving) the graph of the function f by c units upwards.

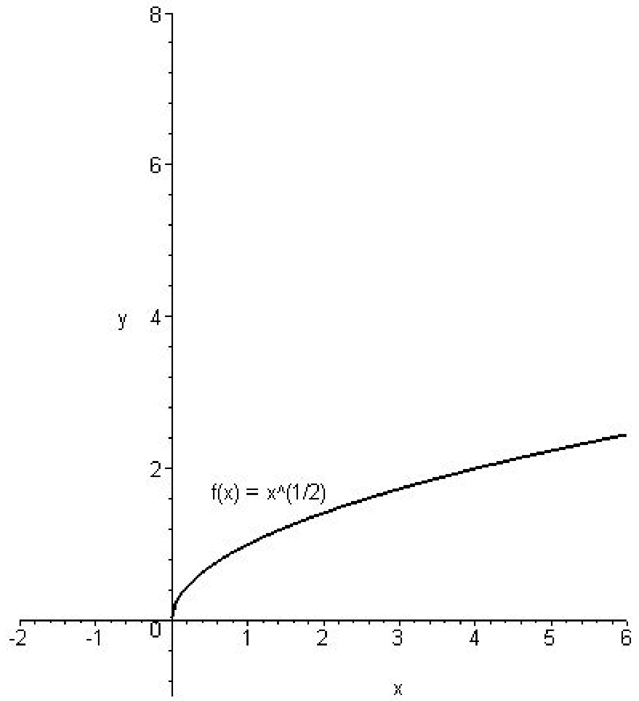
– In the case when c is negative, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = f(x) + c$ by shifting (or moving) the graph of the function f by c units downwards.

– In the case when c is nonnegative, we can obtain the graph of the function $g : D_c \to \mathbb{R}$, $g(x) = f(x + c)$ by shifting (or moving) the graph of the function f by c units to the left.

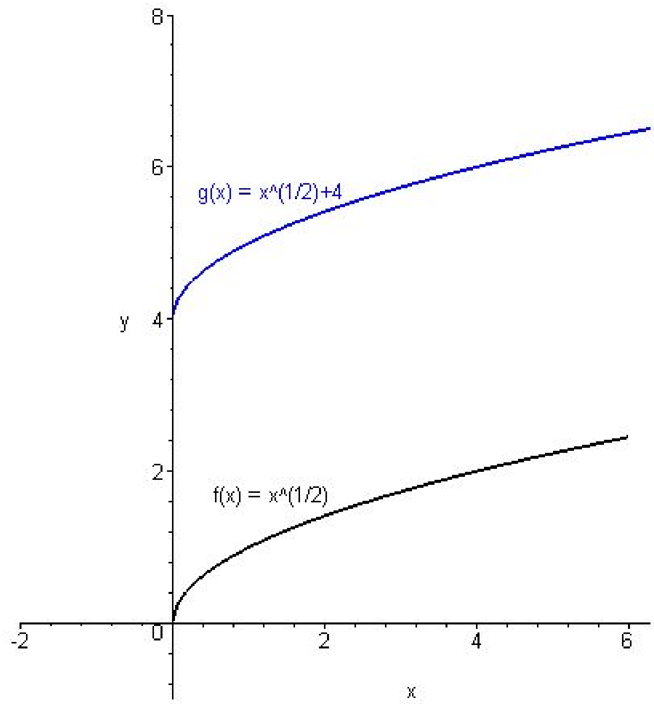
– In the case when c is negative, we can obtain the graph of the function $g : D_c \to \mathbb{R}$, $g(x) = f(x + c)$ by shifting (or moving) the graph of the function f by c units to the right.

Examples.

1. Let us sketch the graph of the function $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{(}x) + 4$ using the graph of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{(}x)$.



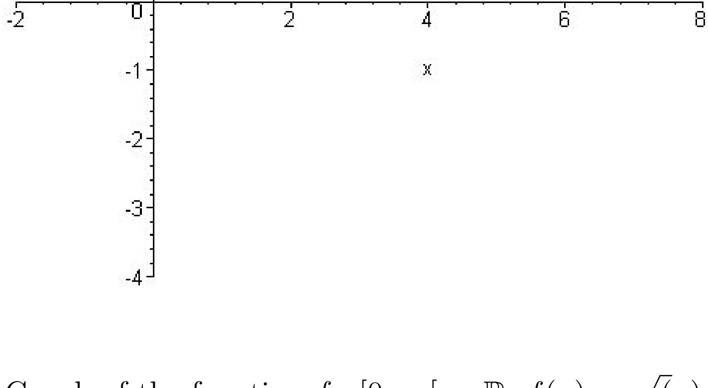
Graph of the function $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{(}x)$.



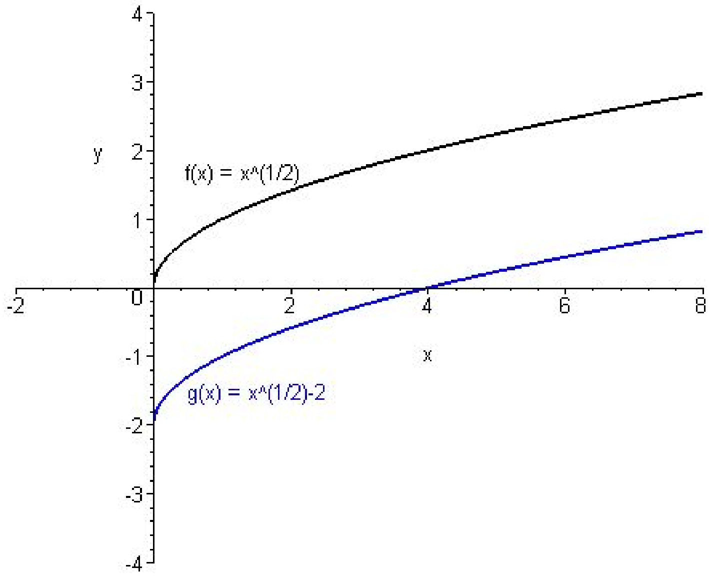
Graphs of the functions $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{(}x)$ and $g : [0, \infty[\to \mathbb{R}$,

$$g(x) = \sqrt{(x)} + 4.$$

2. Let us sketch the graph of the function $g : [0, \infty[\to \mathbb{R}$, $g(x) = \sqrt{(x)} - 2$ using the graph of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{(x)}$.

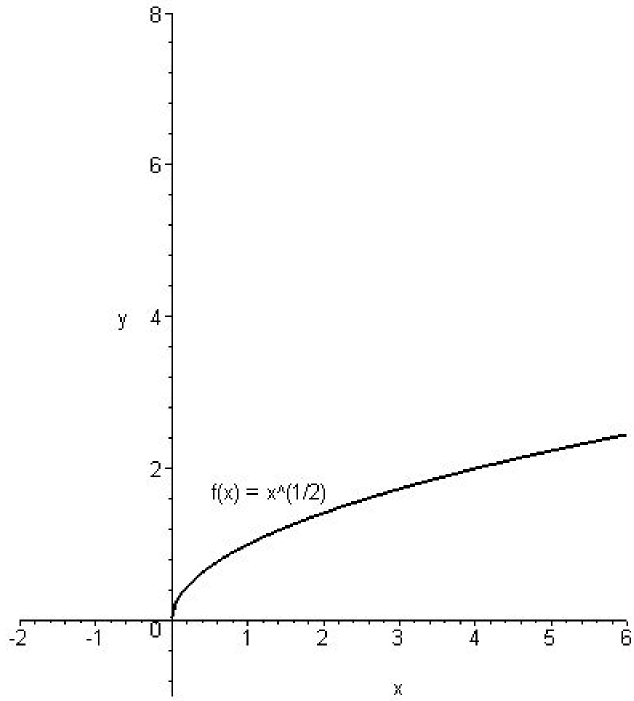


Graph of the function $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{(x)}$.

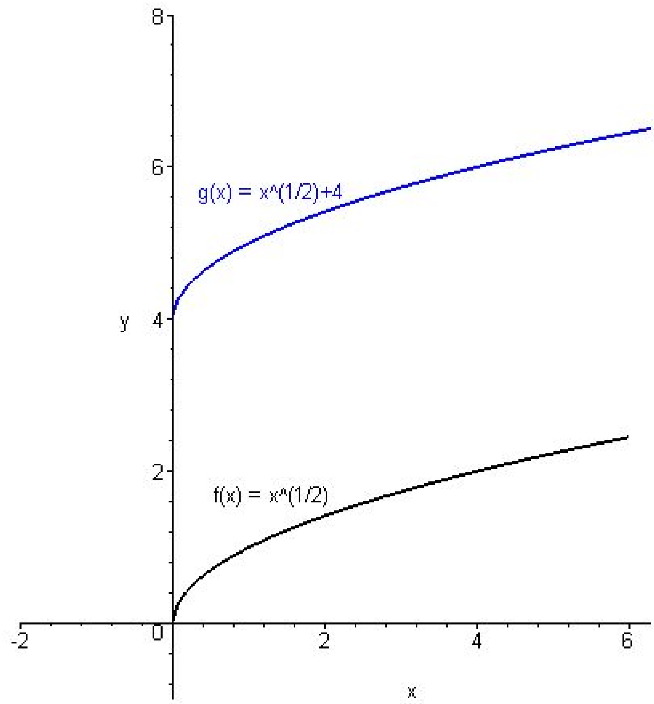


Graphs of the functions $f : [0, \infty[\rightarrow \mathbb{R}$, $f(x) = \sqrt{(}x)$ and $g : [0, \infty[\rightarrow \mathbb{R}$, $g(x) = \sqrt{(}x) - 2$.

3. Let us sketch the graph of the function $g : [0, \infty[\rightarrow \mathbb{R}$, $g(x) = \sqrt{(}x) - 2$ using the graph of $f : [0, \infty[\rightarrow \mathbb{R}$, $f(x) = \sqrt{(}x)$.



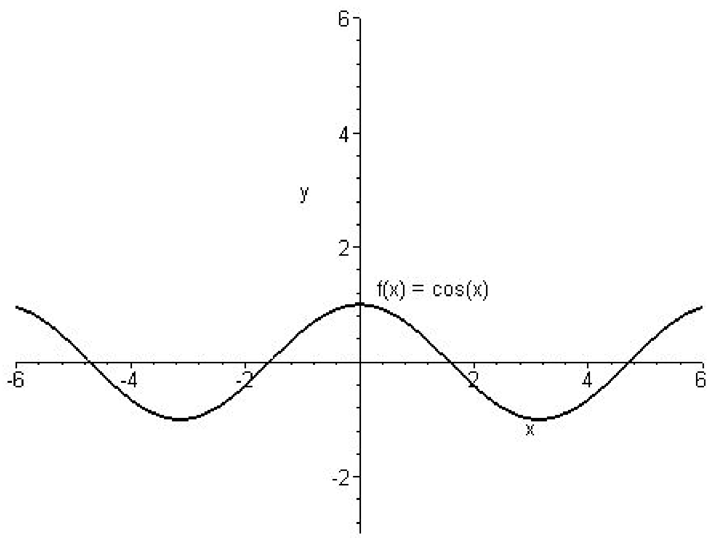
Graph of the function $f : [0, \infty[\to \mathbb{R},\ f(x) = \sqrt{(}x)$.



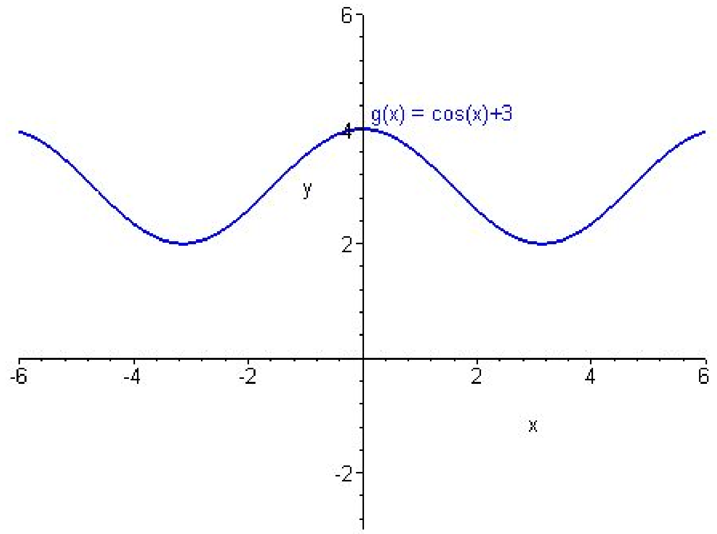
Graphs of the functions $f : [0, \infty[\to \mathbb{R},\ f(x) = \sqrt{(}x)$ and $g : [0, \infty[\to \mathbb{R}$,

$$g(x) = \sqrt{(}x) + 4.$$

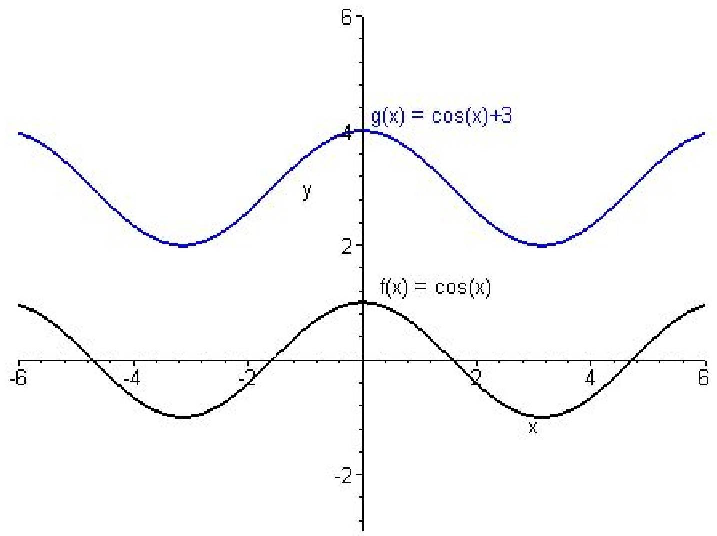
4. Let us sketch the graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(x) + 3$ separately and in the same coordinate system, too.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$.

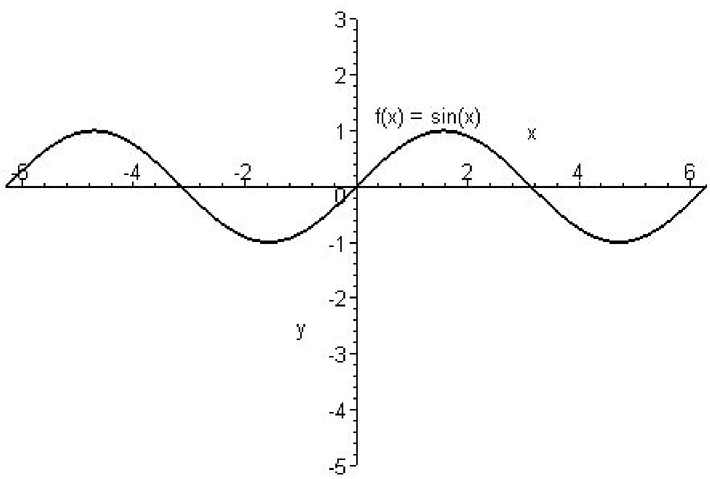


Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(x) + 3$.

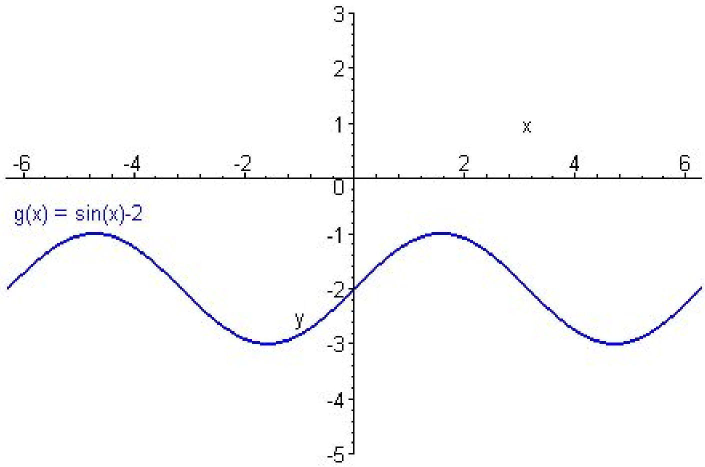


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(x) + 3$.

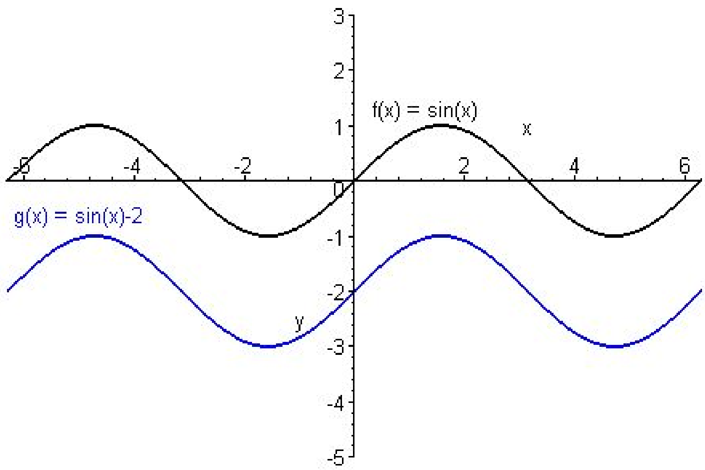
5. Let us sketch the graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(x) - 2$ separately and in the same coordinate system, too.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$.



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(x) - 2$.



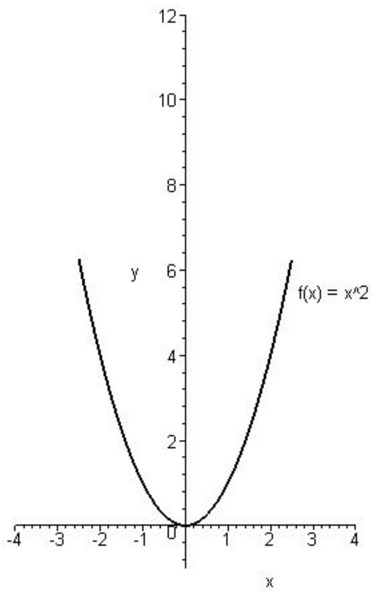
Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(x) - 2$.

4.2 Scaling graphs of functions

In this section, we will study, how we can get the graph of the functions $f(cx)$ and $cf(x)$ from the graph of f, where $c \neq 0$ is a real constant.

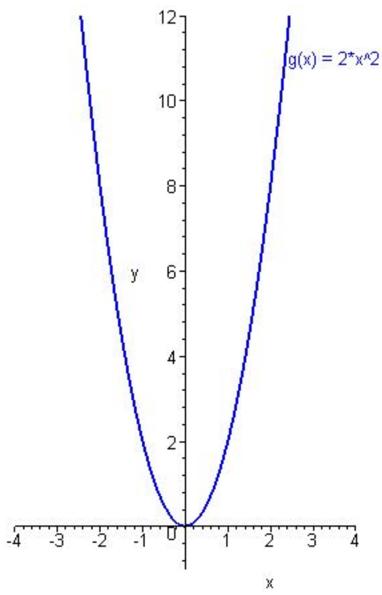
Similarly to the previous part, we try to get some observations based on our knowledge of drawing graphs of functions.

First, we consider the transformation type '$cf(x)$', and we start again with the graph of the simple quadratic function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.



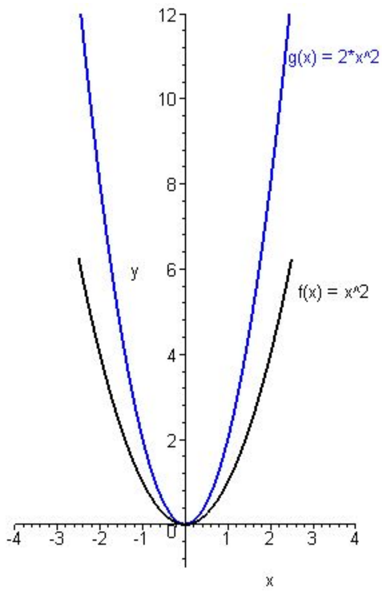
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

Let us consider the graph of $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x^2$:



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x^2$.

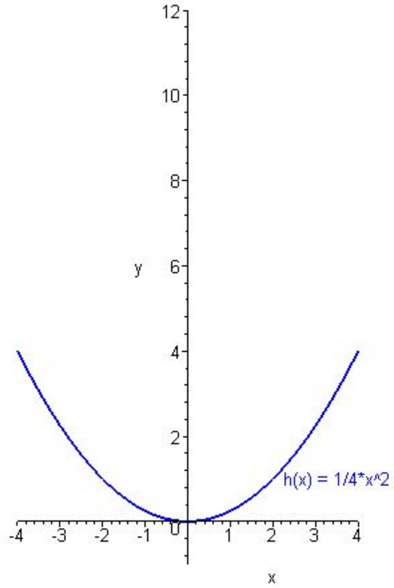
Finally, we sketch both functions in the same coordinate system:



Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x^2$.

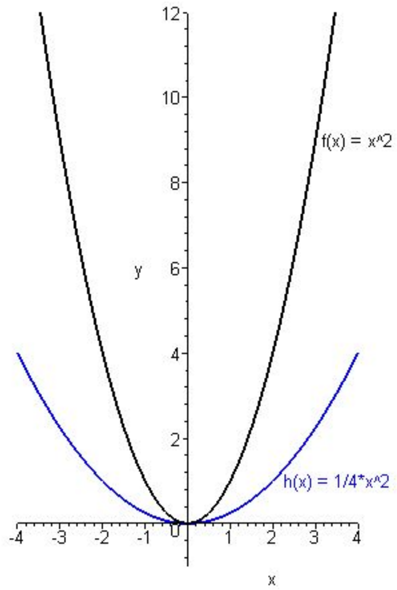
It is easy to see, that, we can get each point (x, z) of the graph of g using $z = 2y$ from the corresponding point (x, y) of the graph f. Since this property is valid for all pairs of points (x, y) and (x, z) of the graphs of our functions, we may obtain the graph of g by stretching (or magnifying) the graph of the function f with the factor c vertically.

Let us investigate the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \frac{1}{4}x^2$ now.



Graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \frac{1}{4}x^2$.

Sketching f and h in the same coordinate system, we obtain:

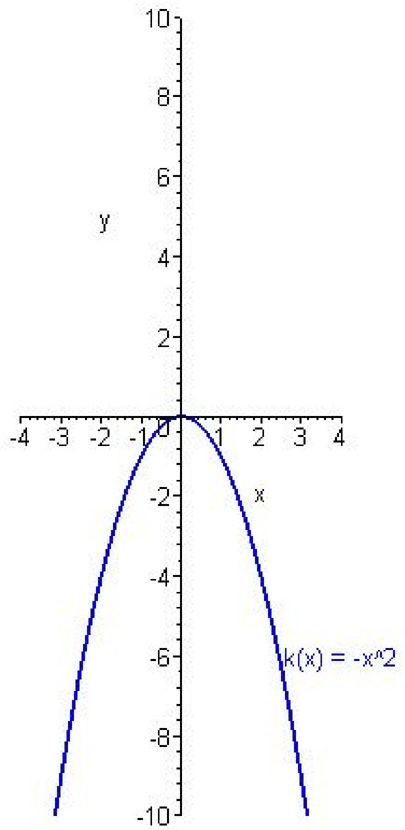


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \frac{1}{2}x^2$.

Here, each point (x, z) of the graph of h can be obtained by the calculation $z = \frac{1}{4}y$ from the corresponding point (x, y) of the graph f. Therefore, we get the graph of the function h by compressing (or shrinking) the graph of the function f with the factor 2 vertically.

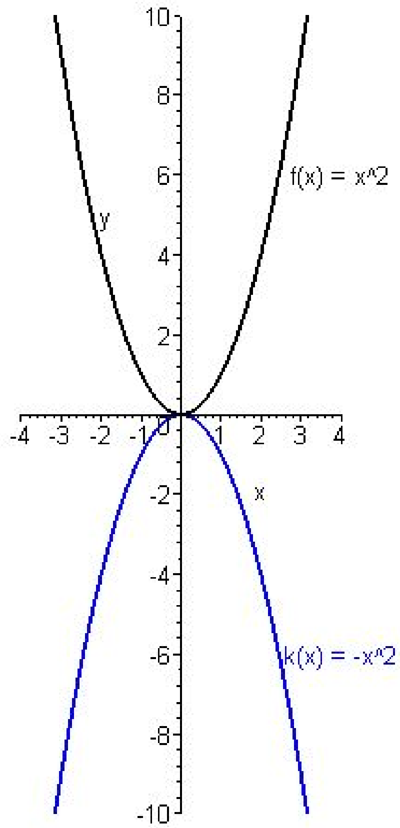
It is important to note, that the sign of c plays also here an important role.

Namely, the graph of $k : \mathbb{R} \to \mathbb{R}$, $k(x) = -x^2$ is:



Graph of the function $k : \mathbb{R} \to \mathbb{R}$, $k(x) = -x^2$.

If we draw it together with the graph of f, we get

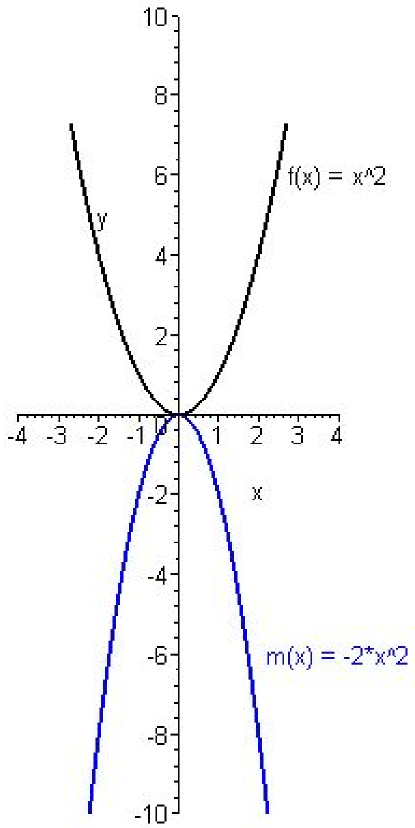


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $k : \mathbb{R} \to \mathbb{R}$, $k(x) = -x^2$.

It is easy to see, that we can get the graph of k by reflecting the graph of f about the x-axis.

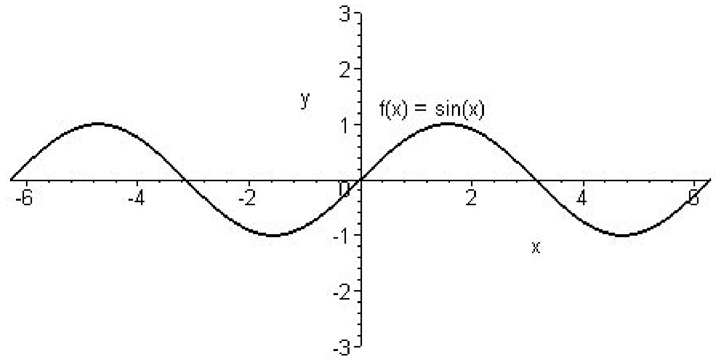
If we would like to determine the graph of the function $m : \mathbb{R} \to \mathbb{R}$, $m(x) = -2x^2$, we have to combine the two steps above: we have to stretch (or magnify) the graph of the function f with the factor c vertically, and we have to reflect this (new) graph about the x-axis.

This situation is illustrated in the following picture.



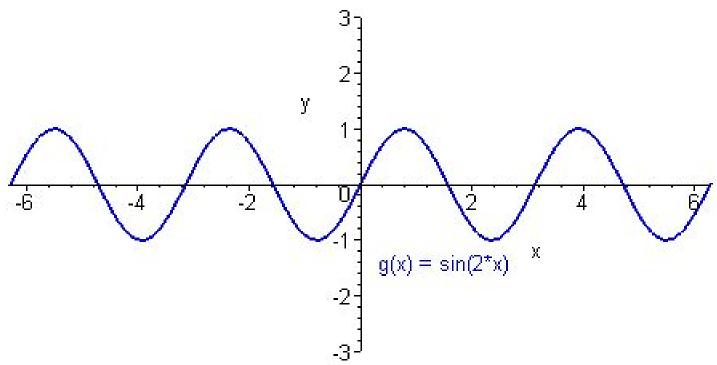
Graphs of the functions $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$ and $m:\mathbb{R}\to\mathbb{R}$, $m(x)=-2x^2$.

Let us investigate now the transformations of the type '$f(cx)$'. In the case of the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$, we have $f(cx)=cf(x)=cx^2$, therefore, it does not give new information for us in this case By this reason, in the following, we will consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin(x)$ us our 'basis function'. The graph of this function is:



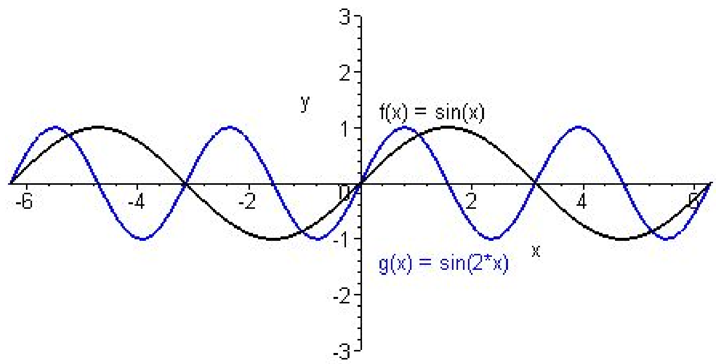
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$.

Let us consider now the graph of $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(2x)$.



Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(2x)$.

Sketching these two graphs in the same coordinate system, we get

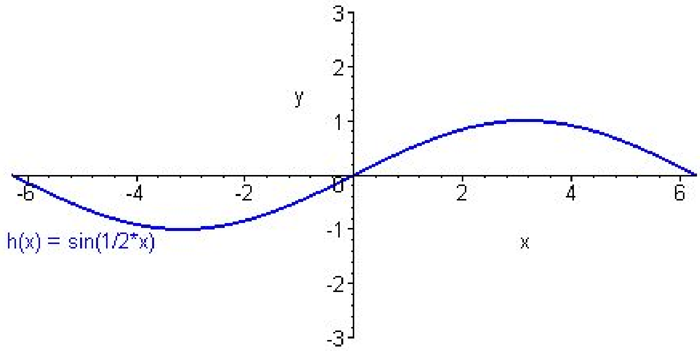


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \sin(2x)$.

It is visible that we can get the graph of g by stretching (or magnifying) the graph of f by the factor 2 horizontally.

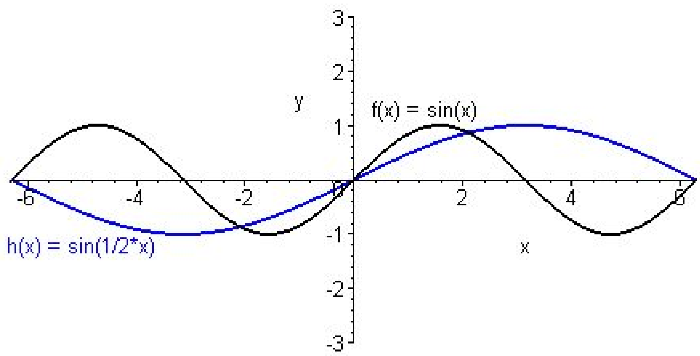
The situation is analogous, if our constant is positive, but less than 1. In this case we compress (or shrinking) the graph instead of stretching (or magnifying) it.

We illustrate these situations with the functions f and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \sin\left(\frac{1}{2}x\right)$. in the following pictures.



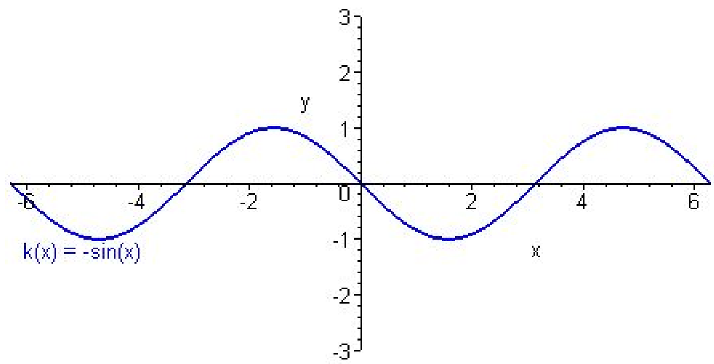
Graph of the function $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \sin\left(\frac{1}{2}x\right)$.

Sketching these two graphs in the same coordinate system, we get

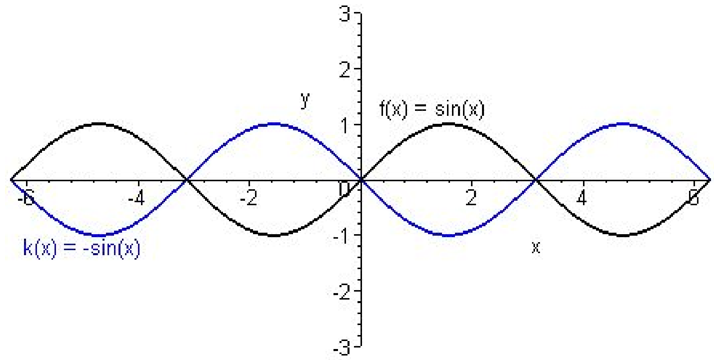


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \sin\left(\frac{1}{2}x\right)$.

Finally, we will investigate the case when our factor c is negative. Let us consider the function $k : \mathbb{R} \to \mathbb{R}$, $k(x) = \sin(-x)$.



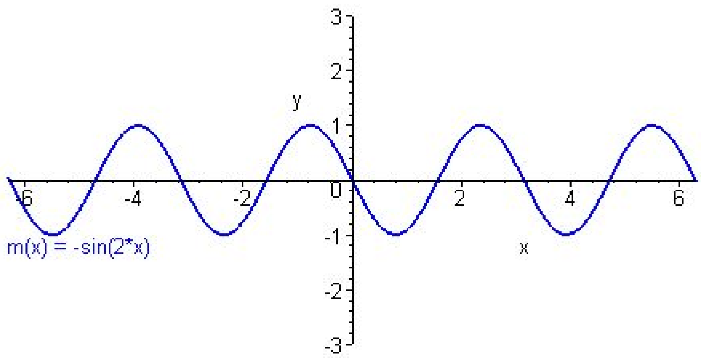
Graph of the function $k : \mathbb{R} \to \mathbb{R}$, $k(x) = \sin(-x)$.



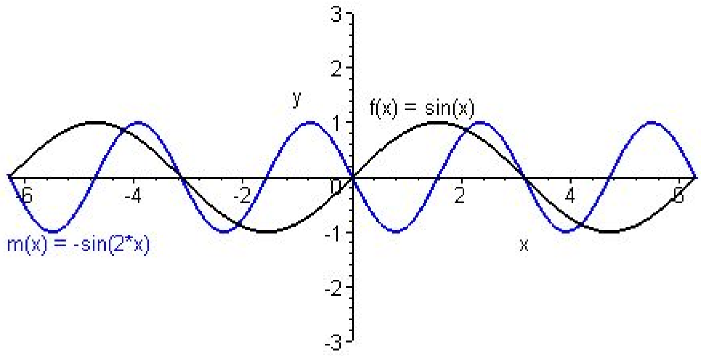
Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $k : \mathbb{R} \to \mathbb{R}$, $k(x) = \sin(-x)$.

Similarly to the case of a 'negative constant c' above, we easily get, that in this case, after stretching (or magnifying) or compressing (or shrinking) the graph of $f(x) = \sin(x)$, we have to reflect the result about the y-axis.

We illustrate this situation with the graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $m : \mathbb{R} \to \mathbb{R}$, $m(x) = \sin(-2x)$.



Graph of the function $m : \mathbb{R} \to \mathbb{R}$, $m(x) = \sin(-2x)$.



Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $m : \mathbb{R} \to \mathbb{R}$, $m(x) = \sin(-2x)$.

It is also here clear that the argumentations above is independent from the concrete functions, it only depends on the connection between them. Thus, we can give a general method for these types of function transformations, too. In order describe this method, let $D \subseteq \mathbb{R}$ be a set, let $c \neq 0$ be a real number and denote $D^{(c)}$ the set $D^{(c)} = \{c \cdot s \mid s \in D\}$.

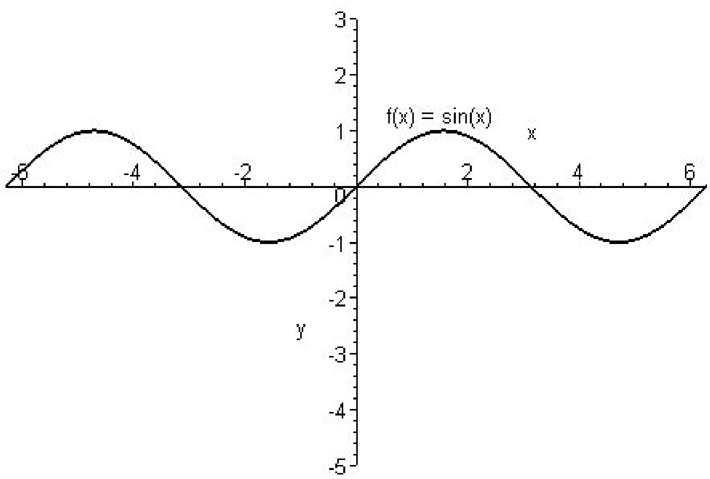
Let $c \neq 0$ be a real number, $D \subseteq \mathbb{R}$ be a set and let us consider a function $f : D \to \mathbb{R}$.

- In the case when $c \geq 1$, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = cf(x)$ by stretching (or magnifying) the graph of the function f with the factor c vertically.
- In the case when $0 < c < 1$, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = cf(x)$ by compressing (or shrinking) the graph of the function f with the factor c vertically.
- In the case when $c \leq -1$, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = cf(x)$ by stretching (or magnifying) the graph of the function f with the factor c vertically, and reflecting this graph about the x-axis.
- In the case when $-1 < c < 0$, we can obtain the graph of the function $g : D \to \mathbb{R}$, $g(x) = cf(x)$ by compressing (or shrinking) the graph of the function f with the factor c vertically, and reflecting this graph about the x-axis.

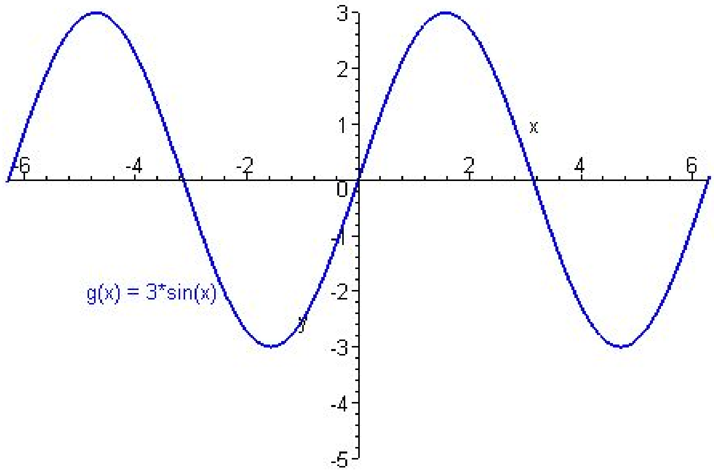
- In the case when $c > 1$, we can obtain the graph of the function $g : D^{(c)} \to \mathbb{R}$, $g(x) = f(cx)$ by stretching (or magnifying) the graph of the function f with the factor c horizontally.
- In the case when $0 < c < 1$, we can obtain the graph of the function $g : D^{(c)} \to \mathbb{R}$, $g(x) = f(cx)$ by compressing (or shrinking) the graph of the function f with the factor c horizontally.
- In the case when $c < -1$, we can obtain the graph of the function $g : D^{(c)} \to \mathbb{R}$, $g(x) = f(cx)$ by stretching (or magnifying) the graph of the function f with the factor c horizontally, and reflecting this graph about the y-axis.
- In the case when $-1 < c < 0$, we can obtain the graph of the function $g : D^{(c)} \to \mathbb{R}$, $g(x) = f(cx)$ by compressing (or shrinking) the graph of the function f with the factor c horizontally, and reflecting this graph about the y-axis.

Examples.

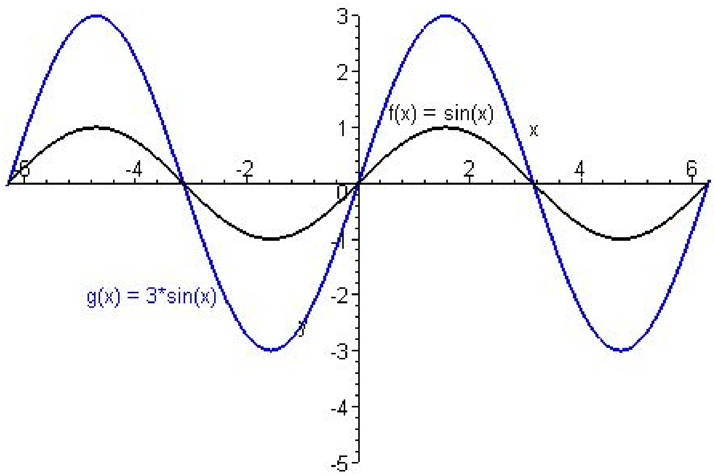
1. Let us sketch the graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3\sin(x)$ separately and in the same coordinate system, too.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$.

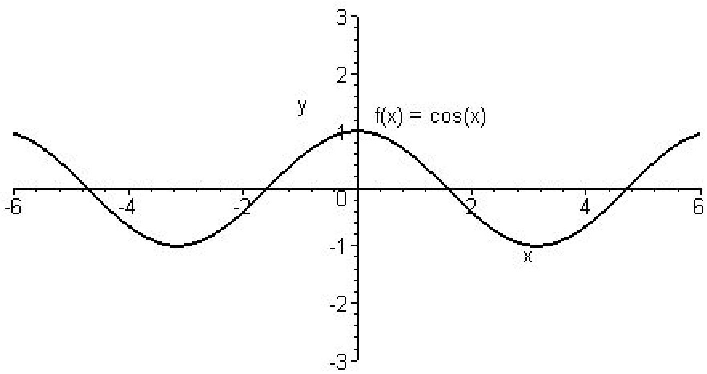


Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3\sin(x)$.

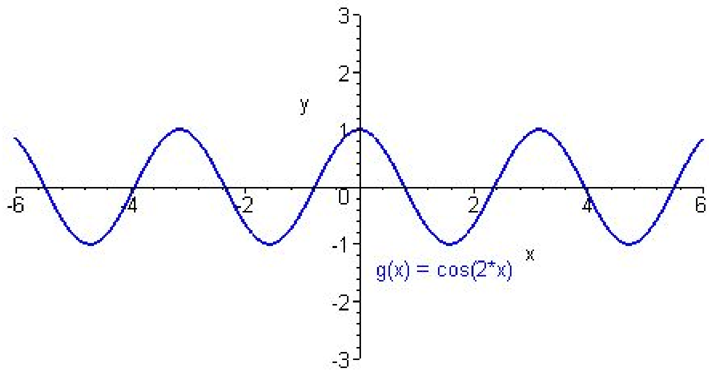


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3\sin(x)$.

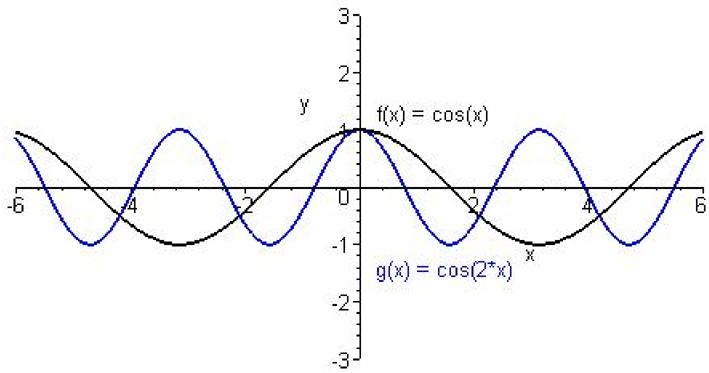
2. Let us sketch the graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(2x)$ separately and in the same coordinate system, too.



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$.

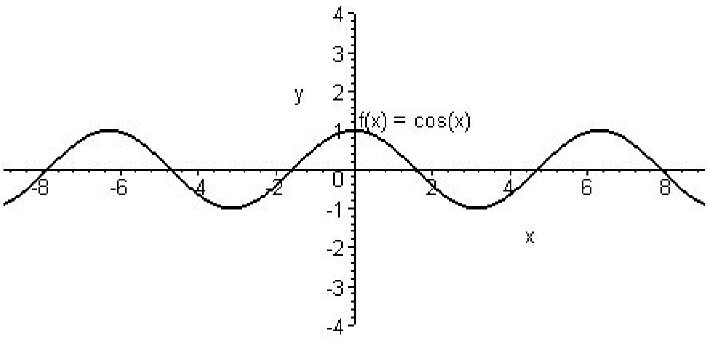


Graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(2x)$.

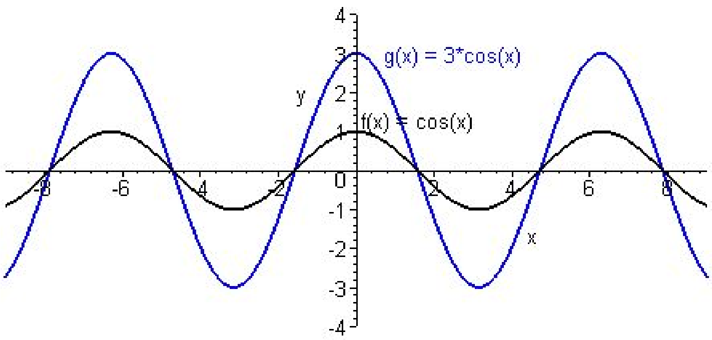


Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \cos(2x)$.

3. Let us sketch the graph of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3\cos(x)$ using the graph of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$.



Graphs of the functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3\cos(x)$.

Chapter 5

Results of the exercises

5.1 Results of the Exercises of Chapter 1

Solution of Exercise 1.1

1). This part of the exercise is trivial, so we do not list the results.

2). The domain of the corresponding function is:

a) $D_f = \mathbb{R}$	b) $D_f = \mathbb{R}$
c) $D_f = [4, \infty[$	d) $D_f =]-\infty, 6]$
e) $D_f = [-2, \infty[$	f) $D_f = \mathbb{R} \setminus \{2\}$
g) $D_f = \mathbb{R} \setminus \{3\}$	h) $D_f = D_f = \mathbb{R} \setminus \{2\}$

3).

a) $y_1 \in R_f,\ y_2 \notin R_f$	b) $y_1 \notin R_f,\ y_2 \in R_f$
c) $y_1 \in R_f,\ y_2 \notin R_f$	d) $y_1 \in R_f,\ y_2 \notin R_f$
e) $y_1 \notin R_f,\ y_2 \in R_f$	f) $y_1 \in R_f,\ y_2 \notin R_f$
g) $y_1 \in R_f,\ y_2 \notin R_f$	h) $y_1 \notin R_f,\ y_2 \in R_f$

4). The range of the corresponding function is:

a) $R_f = \left[-\frac{1}{4}, \infty\right[$ b) $R_f =]-\infty, 4]$

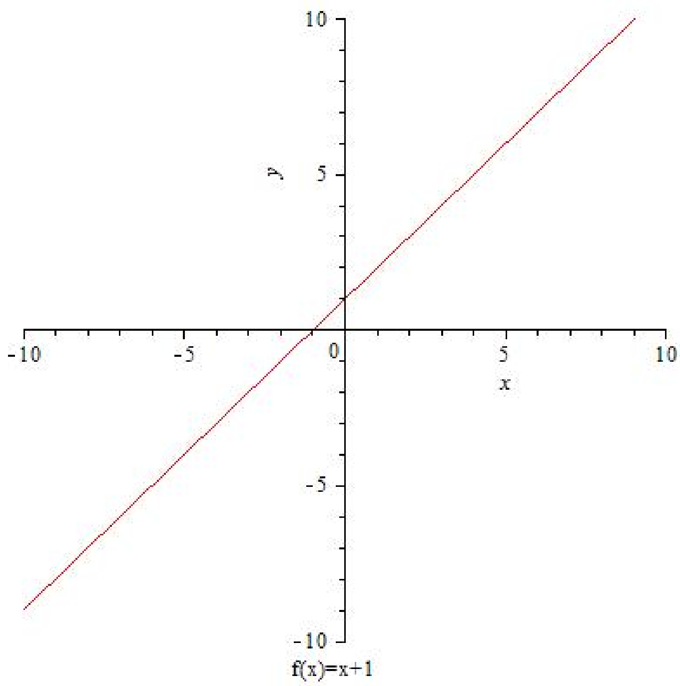
c) $R_f = [0, \infty[$ d) $R_f = [0, \infty[$

e) $R_f = [0, \infty[$ f) $R_f = \mathbb{R} \setminus \{0\}$

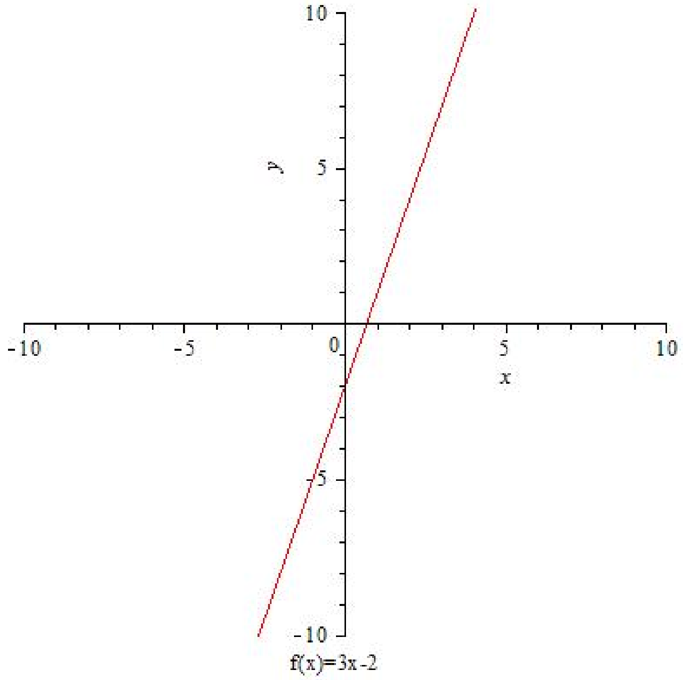
g) $R_f = \mathbb{R} \setminus \{1\}$ h) $R_f = \mathbb{R} \setminus \{2\}$

Solution of Exercise 1.2

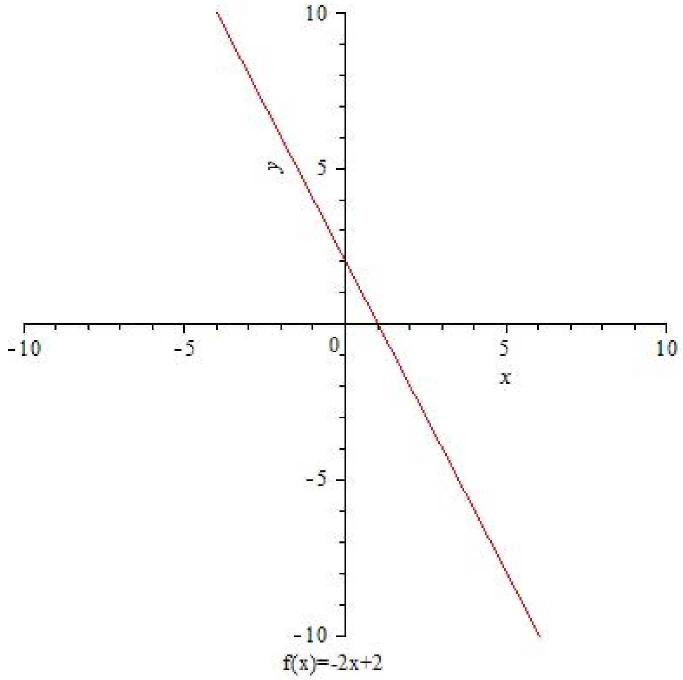
a)



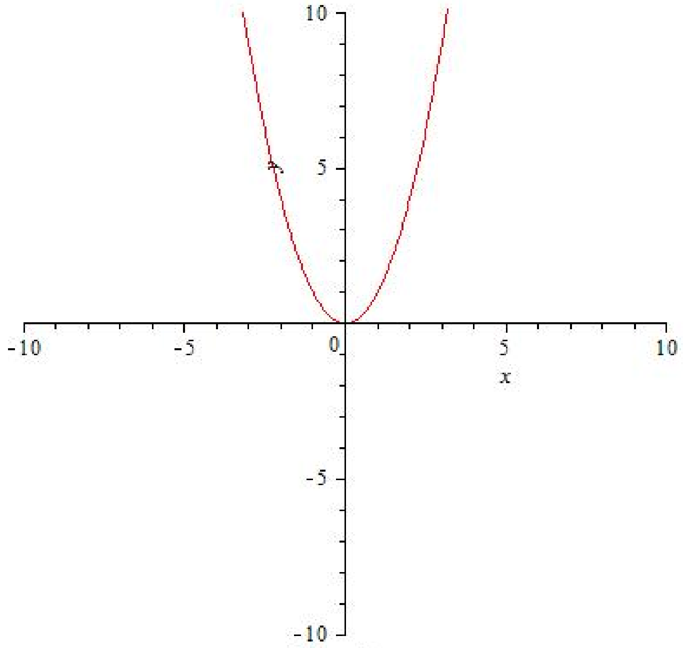
b)



c)

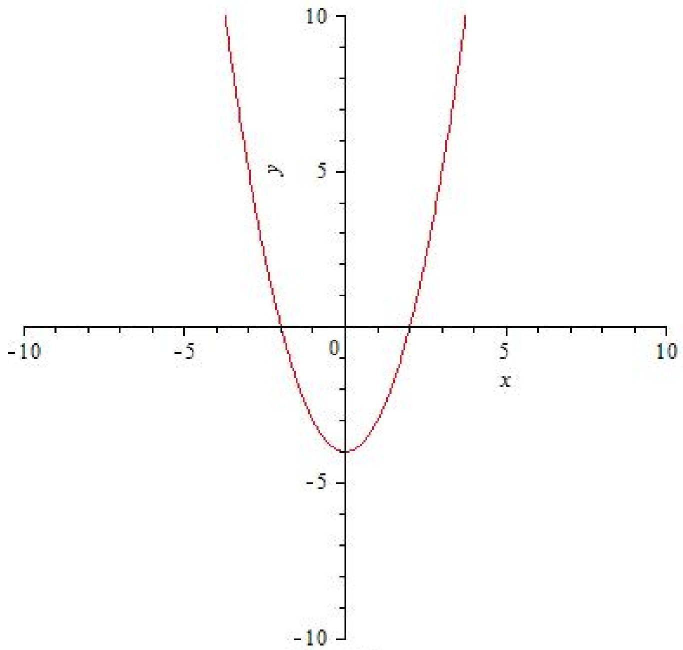


d)



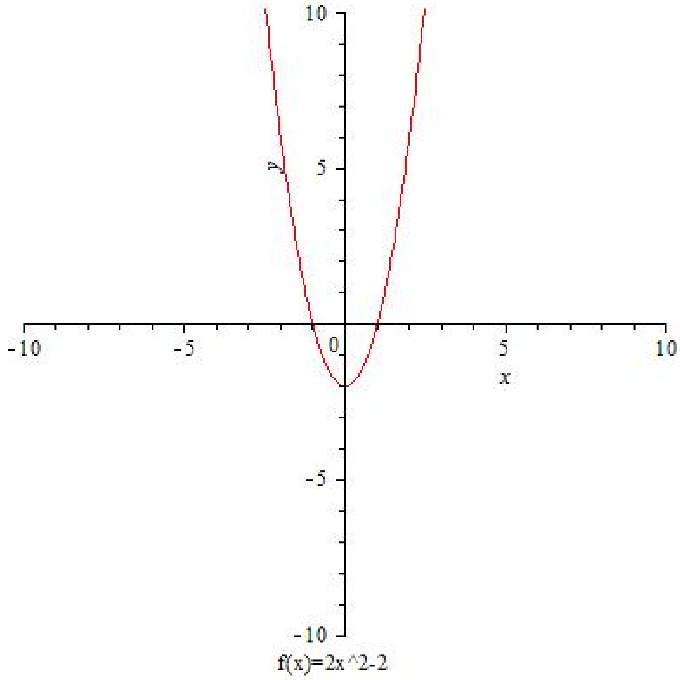
f(x)=x^2

e)

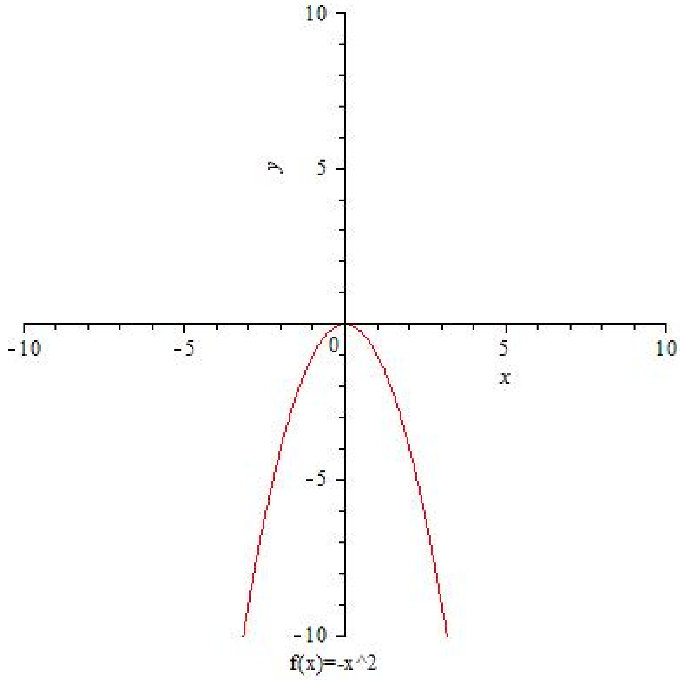


f(x)=x^2-4

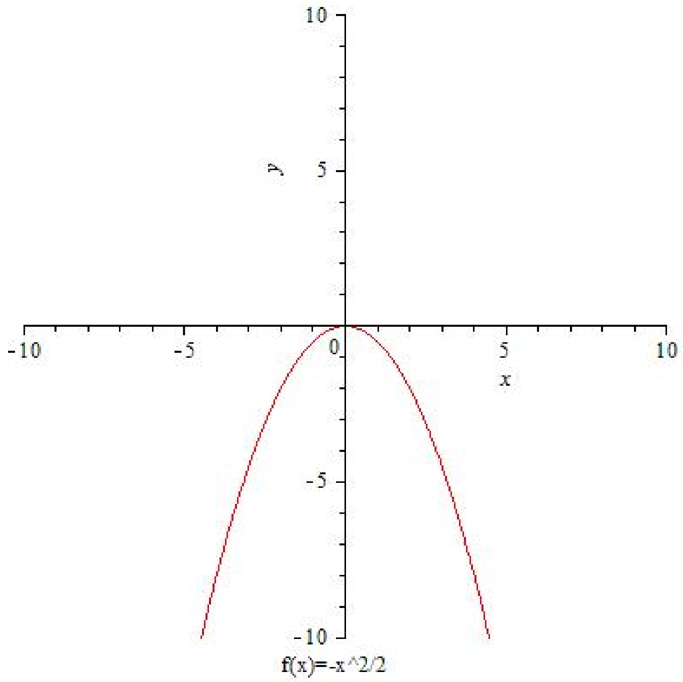
f)



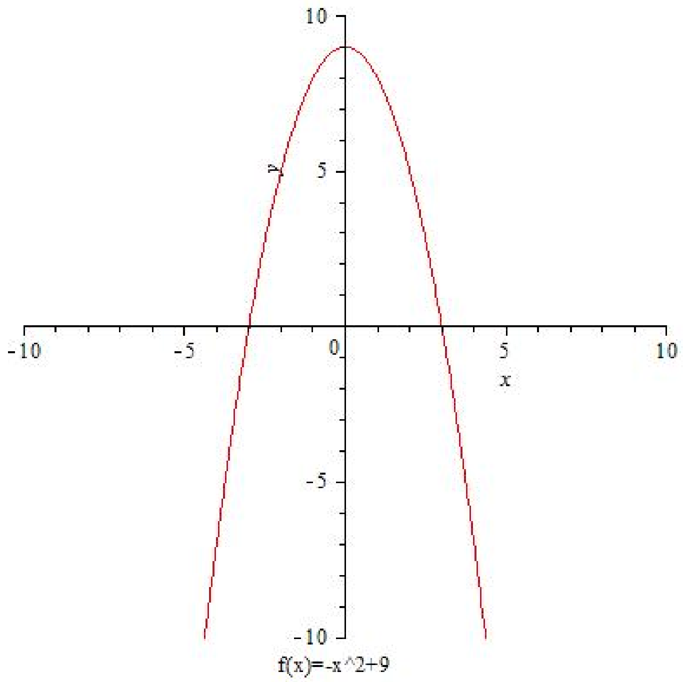
g)



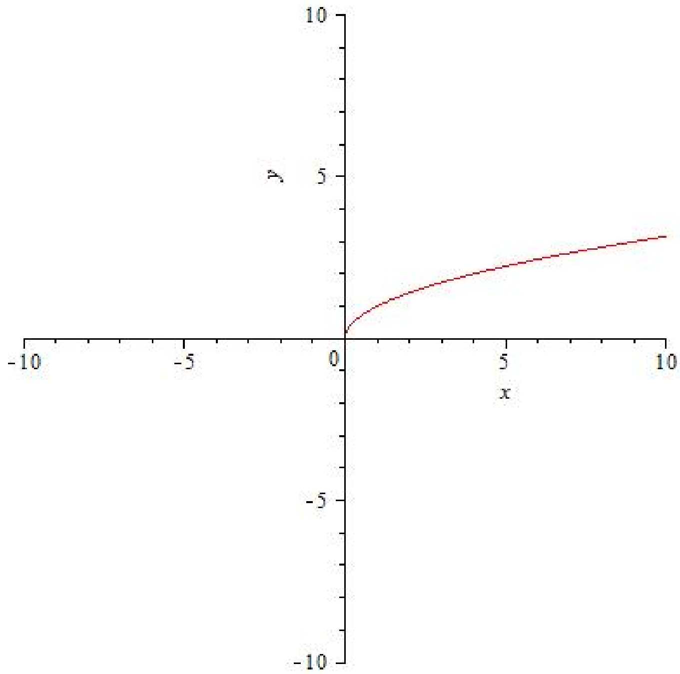
h)



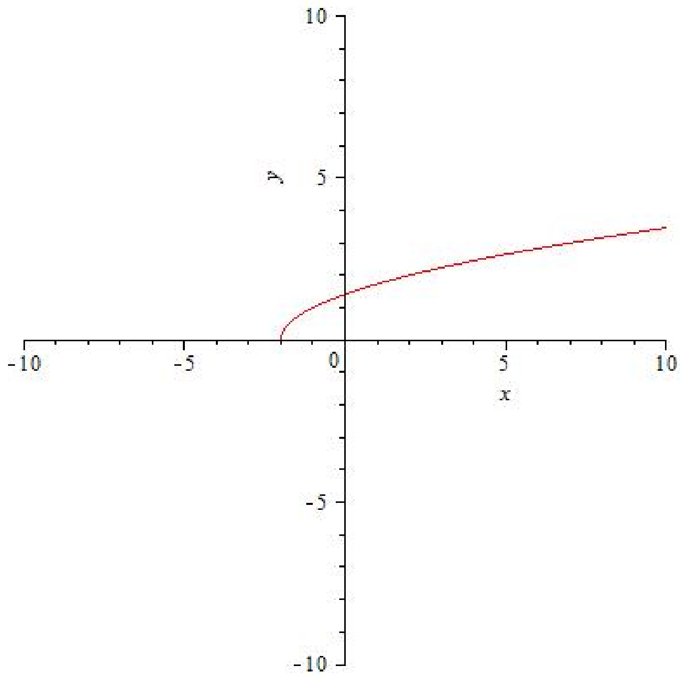
i)



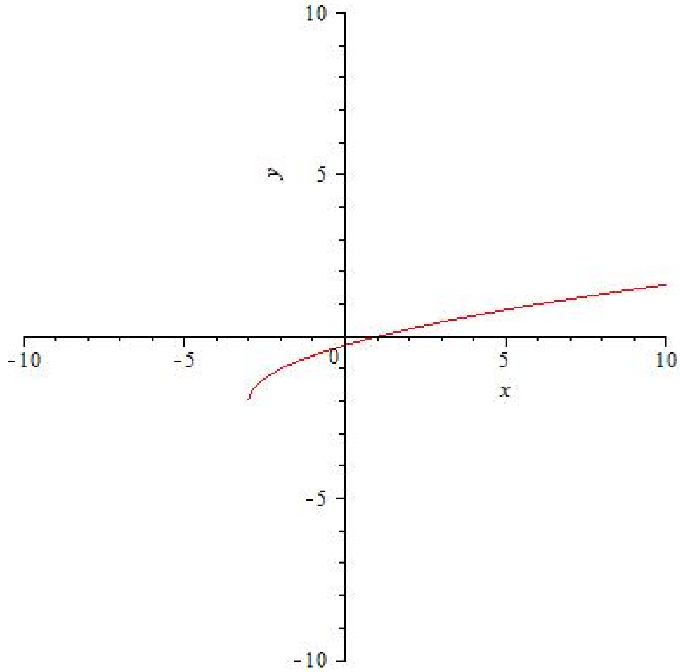
j)



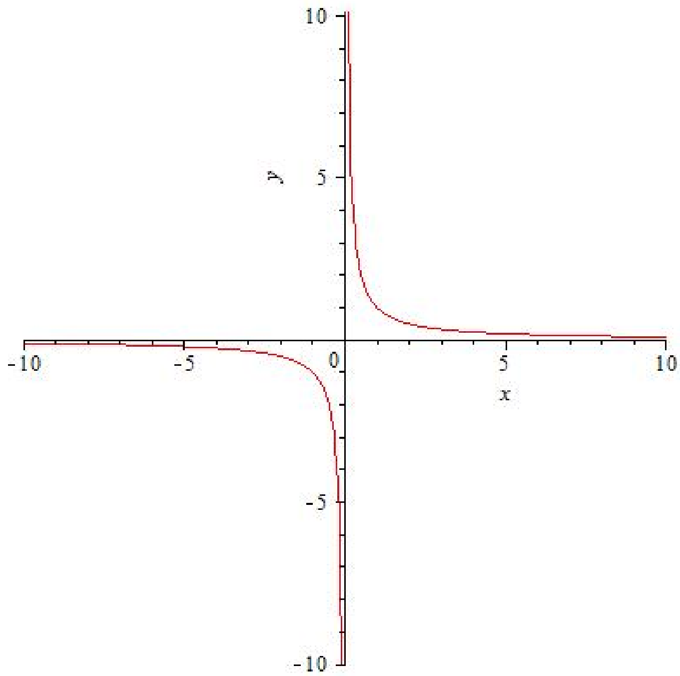
k)



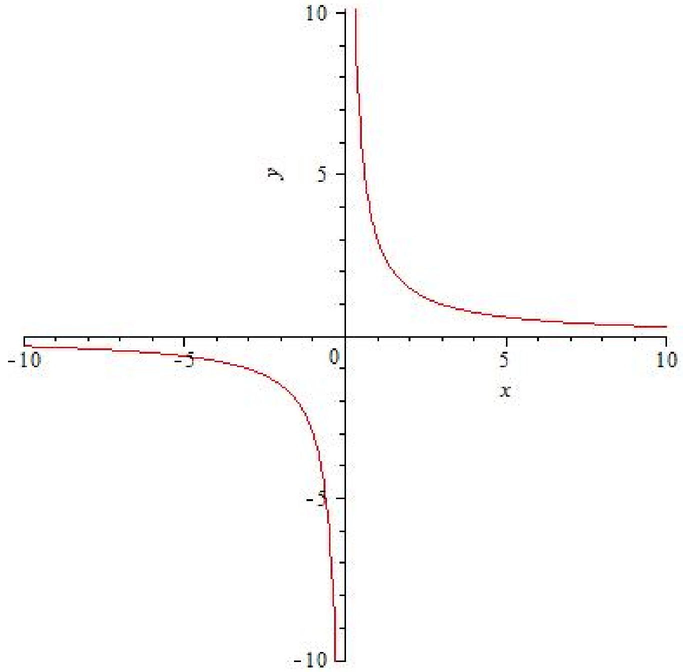
l)



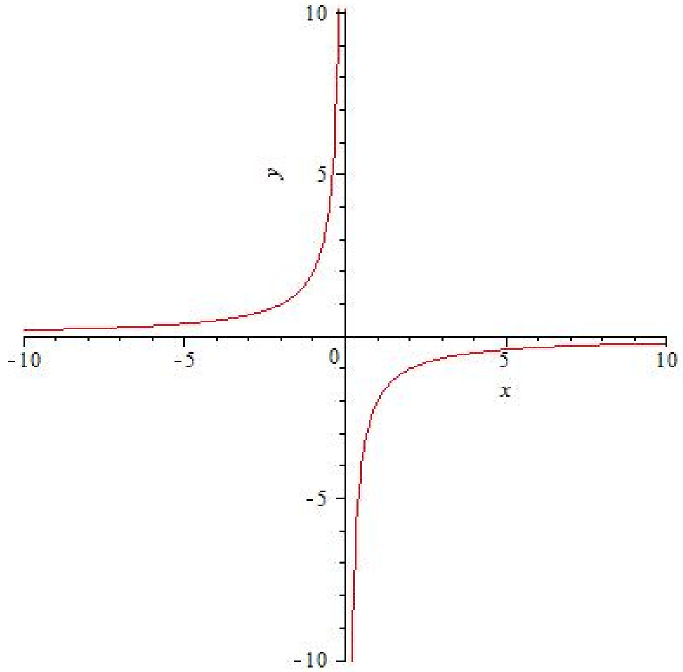
m)



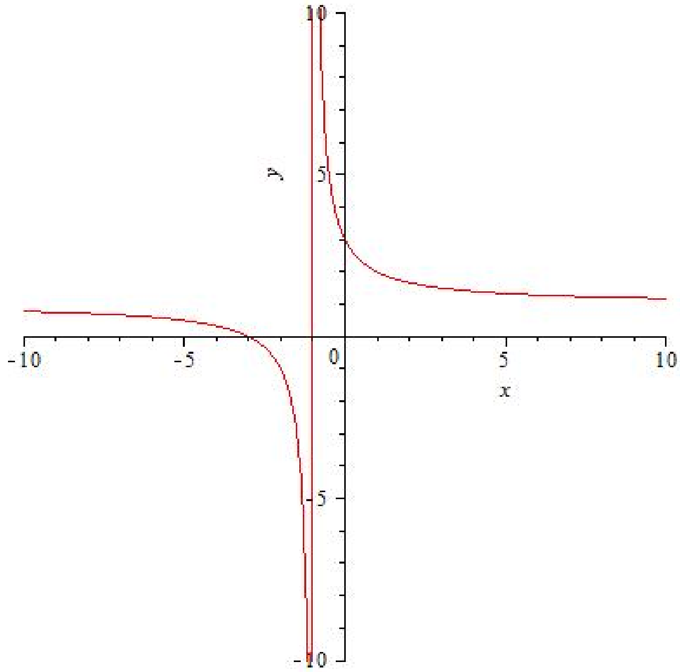
n)



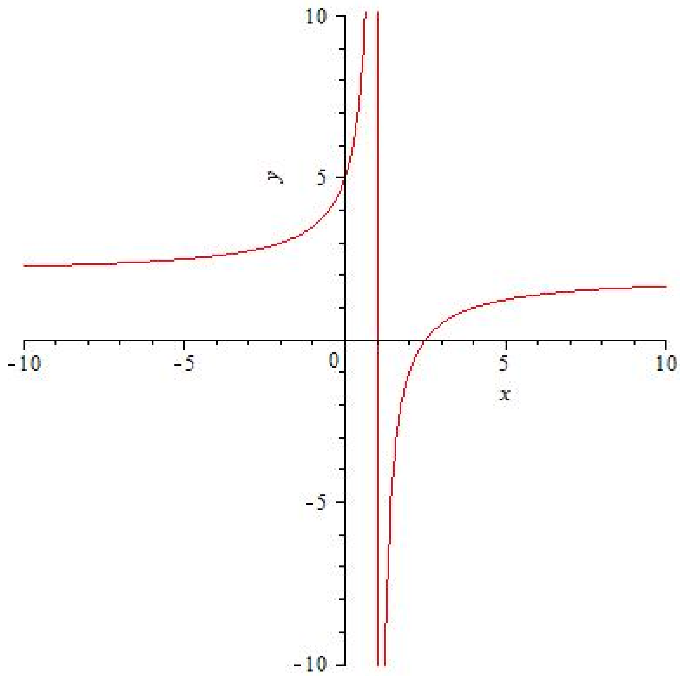
o)



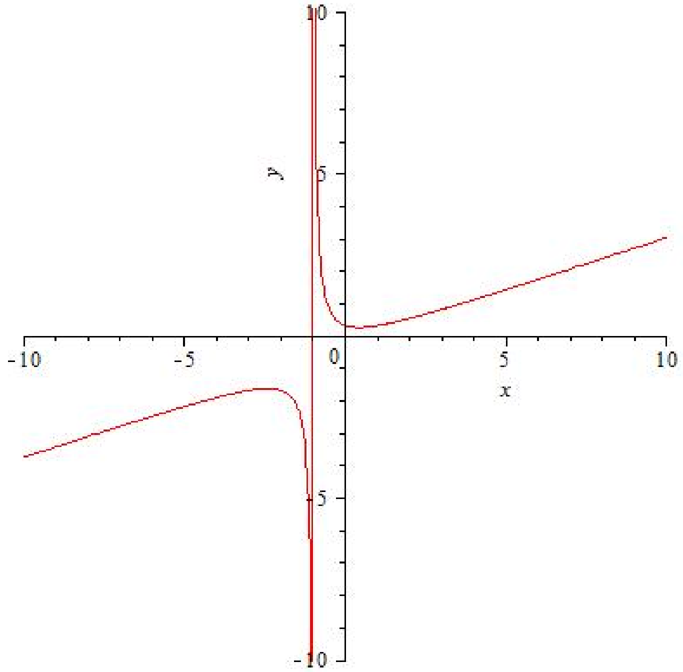
p)



q)

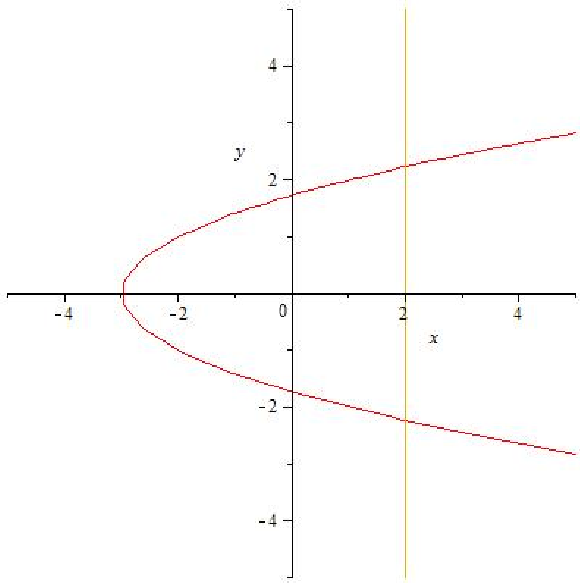


r)

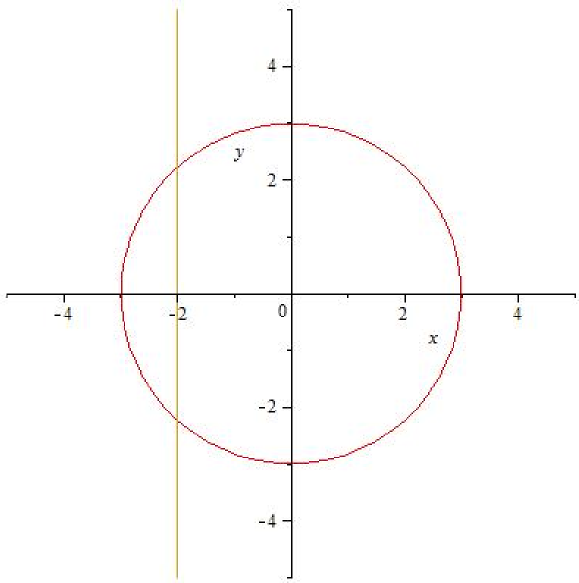


Solution of Exercise 1.3: In the cases a) and c) the curve is not the graph of a function, which is sown by the below application of the vertical line test:

a)



c)



In exercises b) and d) the corresponding curve is the graph of a function.

Solution of Exercise 1.4

The Y-intercept of the corresponding functions is:

a) $(0,-2)$	b) $(0,5)$	c) $(0,3)$
d) $(0,3)$	e) $(0,-2)$	f) $(0,-8)$
g) $(0,)$	h) $(0,)$	i) $\left(0,\dfrac{2}{5}\right)$
j) $(0,0)$	k) $(0,1)$	l) $(0,1)$
m) $(0,0)$	n) $(0,2)$	o) $\left(0,\dfrac{1}{4}\right)$
p) $(0,\sqrt{3})$	q) $(0,3)$	r) $-$
s) $(0,2)$	t) $(0,-2)$	u) $(0,-2)$
v) $(0,0)$	w) $(0,1)$	x) $(0,0)$
y) $-$	z) $(0,2)$	ω) $(0,1)$

The X-intercepts of the corresponding functions are:

a) $\left(\dfrac{2}{3},0\right)$	b) $\left(\dfrac{5}{2},0\right)$	c) $-$
d) $(1,0)\,,\ (3,0)$	e) $\left(\sqrt{2},0\right),\ \left(-\sqrt{2},0\right)$	f) $(2,0)$
g) $(2,0)$	h) $(-1,0)$	i) $(1,0)\,,\ (2,0)$
j) $(0,0)$	k) $-$	l) $-$
m) $(0,0)\,,\ (2,0)\,,(-2,0)$	n) $-$	o) $-$
p) $(-3,0)$	q) $-$	r) $(7,0)$
s) $(2,0)$	t) $(2,0)\,,\ (-4,0)$	u) $(1,0)$
v) $(x_k,0),\ x_k=k\pi,\ k\in\mathbb{Z}$	w) $(x_k,0),\ x_k=\dfrac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$	
x) $(x_k,0),\ x_k=k\pi,\ k\in\mathbb{Z}$	y) $(x_k,0),\ x_k=\dfrac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$	
z) $-$	ω) $(x_k,0),\ x_k=\dfrac{\pi}{4}+k\pi,\ k\in\mathbb{Z}$	

Solution of Exercise 1.5:

a) $(f \circ g)(x) = x^6 + 6x^5 + 12x^4 + 8x^3 + x^2 + 2x + 1$
$(g \circ f)(x) = x^6 + 2x^4 + 4x^3 + x^2 + 4x + 3$
$(f \circ h)(x) = 27x^3 + 54x^2 + 39x + 11$
$(h \circ f)(x) = 3x^3 + 3x + 5$
$(f \circ h \circ g)(x) = 27x^6 + 162x^5 + 378x^4 + 432x^3 + 255x^2 + 78x + 11$
$(g \circ g \circ g)(x) = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x$

b) $(f \circ g)(x) = x^9 - 3x^6 + 3x^3$
$(g \circ f)(x) = x^9 + 3x^6 + 3x^3$
$(f \circ h)(x) = \sin^3 x + 1$
$(h \circ f)(x) = \sin(x^3 + 1)$
$(f \circ h \circ g)(x) = \sin^3(x^3 - 1) + 1$
$(g \circ g \circ g)(x) = x^{27} - 9x^{24} + 36x^{21} - 87x^{18} + 144x^{15} - 171x^{12} + 147x^9 - 90x^6 + 36x^3 - 9$

c) $(f \circ g)(x) = 2^{2x} + 2^x + 1$
$(g \circ f)(x) = 2^{x^2+x+1}$
$(f \circ h)(x) = \tan^2 x + \tan x + 1$
$(h \circ f)(x) = \tan(x^2 + x + 1)$
$(f \circ h \circ g)(x) = \tan^2(2^x) + \tan(2^x) + 1$
$(g \circ g \circ g)(x) = 2^{2^{2^x}}$

d) $(f \circ g)(x) = 2x + \sqrt{2x}$
$(g \circ f)(x) = \sqrt{2x^2 + 2x}$
$(f \circ h)(x) = x^2 - 3x + 2$
$(h \circ f)(x) = x^2 + x - 2$
$(f \circ h \circ g)(x) = 2x - 3\sqrt{2x} + 2$
$(g \circ g \circ g)(x) = \sqrt[8]{128x}$

e) $(f \circ g)(x) = 3^{3x}$
$(g \circ f)(x) = 3^{x^3}$
$(f \circ h)(x) = 27x^3 + 54x^2 + 36x + 8$
$(h \circ f)(x) = 3x^3 + 2$
$(f \circ h \circ g)(x) = 273^{3x} + 543^{2x} + 363^x + 8$
$(g \circ g \circ g)(x) = 3^{3^{3^x}}$

f) $(f \circ g)(x) = \sin^4 x$
$(g \circ f)(x) = \sin x^4$
$(f \circ h)(x) = \log_2^4 x$
$(h \circ f)(x) = \log_2 x^4$
$(f \circ h \circ g)(x) = \log_2^4(\sin x)$
$(g \circ g \circ g)(x) = \sin(\sin(\sin x))$

g) $(f \circ g)(x) = x^6 - 6x^4 + 11x^2 - 5$
$(g \circ f)(x) = x^6 - 2x^4 + 2x^3 + x^2 - 2x - 1$
$(f \circ h)(x) = 8x^3 + 12x^2 + 4x + 1$
$(h \circ f)(x) = 2x^3 - 2x + 3$
$(f \circ h \circ g)(x) = 8x^6 - 36x^4 + 52x^2 - 23$
$(g \circ g \circ g)(x) = x^8 - 8x^6 + 20x^4 - 16x^2 + 2$

h) $(f \circ g)(x) = x^6 - 3x^5 + 6x^4 - 7x^3 + 4x^2 - x$
$(g \circ f)(x) = x^6 - 4x^4 + x^3 + 4x^2 - 2x + 1$
$(f \circ h)(x) = x^6 + 3x^4 + x^2$
$(h \circ f)(x) = x^6 - 4x^4 + 2x^3 + 4x^2 - 4x + 2$
$(f \circ h \circ g)(x) = x^{12} - 6x^{11} + 21x^{10} - 50x^9 + 93x^8 - 138x^7 + 171x^6 - 174x^5 + 148x^4 - 100x^3 + 54$
$(g \circ g \circ g)(x) = x^8 - 4x^7 + 8x^6 - 10x^5 + 9x^4 - 6x^3 + 3x^2 - x + 1$

i) $(f \circ g)(x) = x^6 + x^3 + 3$
$(g \circ f)(x) = x^6 - 3x^5 + 12x^4 - 19x^3 + 36x^2 - 27x + 28$
$(f \circ h)(x) = x^4 + 2x^3 + 2x^2 + x + 3$
$(h \circ f)(x) = x^4 - 2x^3 + 8x^2 - 7x + 13$
$(f \circ h \circ g)(x) = x^{12} + 6x^9 + 14x^6 + 15x^3 + 9$
$(g \circ g \circ g)(x) = x^{27} + 9x^{24} + 36x^{21} + 87x^{18} + 144x^{15} + 171x^{12} + 147x^9 + 90x^6 + 36x^3 + 9$

g) $(f \circ g)(x) = x^4 + 4x^2 + 5$
$(g \circ f)(x) = x^4 - 4x^3 + 8x^2 - 8x + 7$
$(f \circ h)(x) = x^6 + 2x^4 - 2x^3 + x^2 - 2x + 2$
$(h \circ f)(x) = x^6 - 6x^5 + 18x^4 - 32x^3 + 37x^2 - 26x + 10$
$(f \circ h \circ g)(x) = x^{12} + 18x^{10} + 137x^8 + 562x^6 + 1306x^4 + 1624x^2 + 842$
$(g \circ g \circ g)(x) = x^8 + 12x^6 + 60x^4 + 144x^2 + 147$

g) $(f \circ g)(x) = x^6 + 3x^4 + x^2 - 1$
$(g \circ f)(x) = x^6 - 4x^4 + 4x^2 + 1$
$(f \circ h)(x) = x^6 + 3x^5 + 3x^4 + x^3 - 2x^2 - 2x$
$(h \circ f)(x) = x^6 - 4x^4 + x^3 + 4x^2 - 2x$
$(f \circ h \circ g)(x) = x^{12} + 9x^{10} + 33x^8 + 63x^6 + 64x^4 + 30x^2 + 4$
$(g \circ g \circ g)(x) = x^8 + 4x^6 + 8x^4 + 8x^2 + 5$

g) $(f \circ g)(x) = x^6 - 3x^5 + 5x^3 + x^2 - 4x - 1$
$(g \circ f)(x) = x^6 + 2x^4 + x^3 + x^2 + x - 1$
$(f \circ h)(x) = x^6 + 3x^5 + 6x^4 + 7x^3 + 7x^2 + 4x + 3$
$(h \circ f)(x) = x^6 + 2x^4 + 3x^3 + x^2 + 3x + 3$
$(f \circ h \circ g)(x) = x^{12} - 6x^{11} + 12x^{10} - 5x^9 - 9x^8 + 15x^6 - 8x^4 - 7x^3 + 3x^2 + 4x + 3$
$(g \circ g \circ g)(x) = x^8 - 4x^7 + 14x^5 - 7x^4 - 14x^3 + 7x^2 + 3x - 1$

Solution of Exercise 1.6

a) $F(x) = f_4\left(c\left(g_7\left(l_3\left(t\left(h_5(x)\right)\right)\right)\right)\right)$

b) $F(x) = f_7\left(l_3\left(f_2\left(s\left(g_7\left(l_3\left(t\left(h_5(x)\right)\right)\right)\right)\right)\right)\right)$

c) $F(x) = t\left(f_4\left(l_2\left(\cos\left(g_4\left(l_5\left(s\left(h_3(x)\right)\right)\right)\right)\right)\right)\right)$

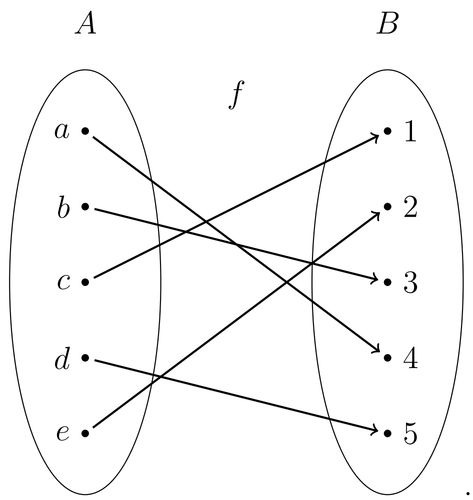
d) $F(x) := f_3\left(s\left(f_4\left(l_2\left(t\left(g_7\left(s\left(f_5(x)\right)\right)\right)\right)\right)\right)\right)$

e) $F(x) = f_5\left(l_3\left(f_2\left(s\left(g_6\left(c\left(h_{11}\left(l_4(x)\right)\right)\right)\right)\right)\right)\right)$

f) $F(x) = f_2\left(s\left(f_5\left(t\left(f_3\left(l_2\left(c\left(h_7\left(f_2(x)\right)\right)\right)\right)\right)\right)\right)\right)$

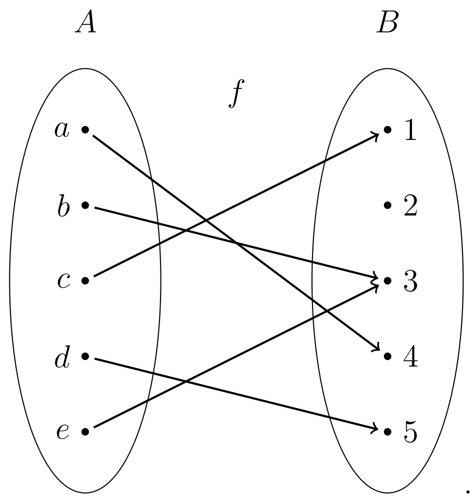
Solution of Exercise 1.7

a) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 5$, $f(e) = 2$. We can illustrate f in the following form:



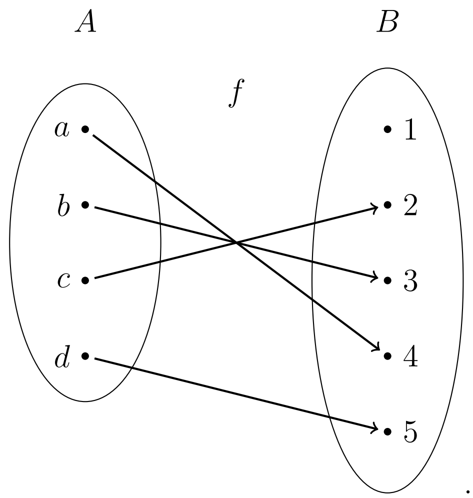
It is easy to see that f is a function. It is also visible that there are no elements $x_1 \in A$ and $x_2 \in A$ such that $f(x_1) = f(x_2)$, which implies that f is injective. Since, for each element $y \in B$, there exists an $x \in A$ such that $f(x) = y$, the function f is also surjective. These properties imply that it is also bijective.

b) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 5$, $f(e) = 3$. We can illustrate f in the following form:



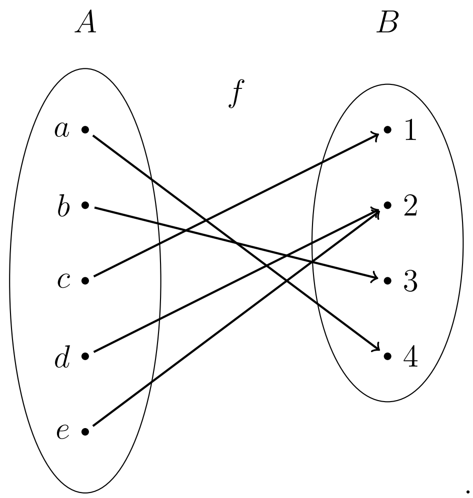
It is obvious that f is a function. Since $f(2) = f(5) = 3$, f is not injective. Furthermore, there is no $x \in A$ for which $f(x) = 2$, thus f is not surjective. This means that f is not bijective.

c) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 2$, $f(d) = 5$. We can illustrate f in the following form:



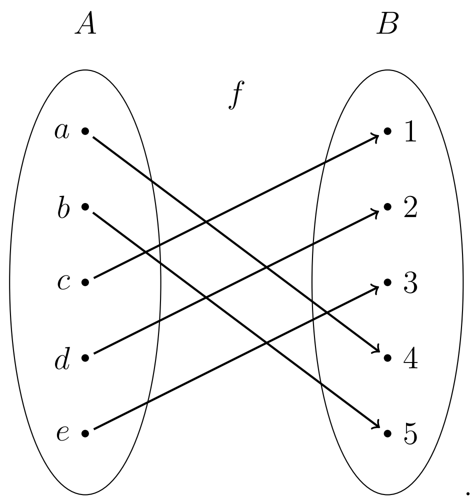
It is easy to se that f is a function and it is injective. However, since there is no $x \in A$ for which $f(x) = 2$, the function f is not surjective. This implies that f is not bijective.

d) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 1$, $f(d) = 2$, $f(e) = 2$. We can illustrate f in the following form:



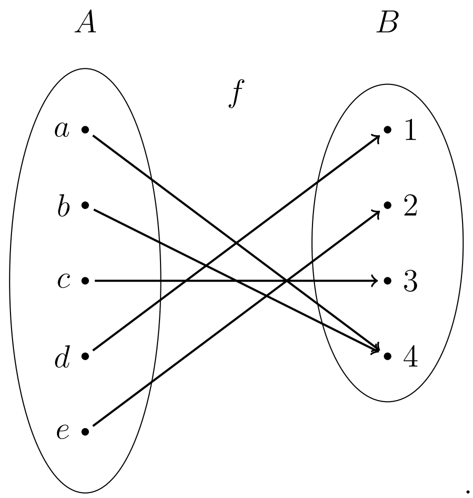
It is visible that f is a function and it is surjective. However f is not injective, therefore, it is not bijective.

e) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 2$, $f(e) = 3$. We can illustrate f in the following form:



It is easy to see that f is a function, it is injective, surjective, which implies that it is also bijective.

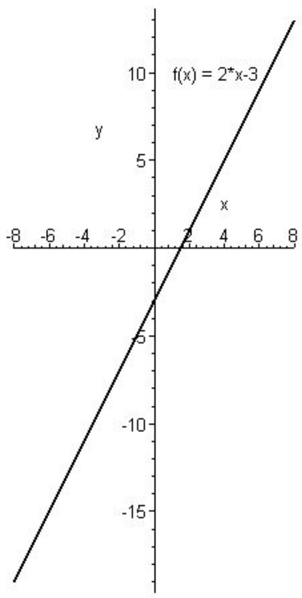
f) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 4$, $f(c) = 3$, $f(d) = 1$, $f(e) = 2$. We can illustrate f in the following form:



It is visible that f is a surjective function. However f is not injective, therefore, it is not bijective.

Solution of Exercise 1.8

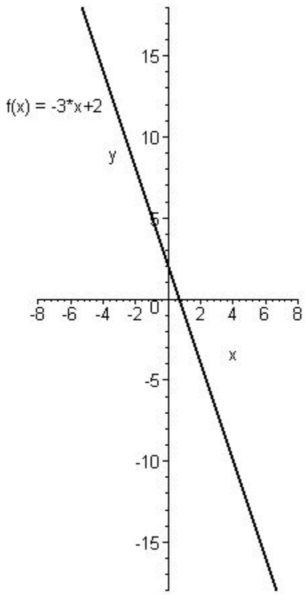
a) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$. The graph of f is:



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 3$.

It is easy to see that there are no elements x_1 and x_2 in the domain $\mathbb{R}$ of the function f such that $f(x_1) = f(x_2)$, which implies that f is injective. It is also clear that, for each element y of the range $\mathbb{R}$ of the function f, there exists an x in the domain $\mathbb{R}$ of f such that $f(x) = y$, therefore, f is surjective. These properties imply that f is bijective.

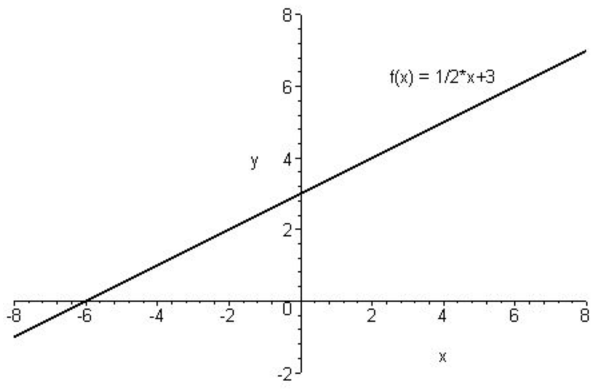
b) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3x + 2$. The graph of f is:



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3x + 2$.

It is easy to see that, similarly to the previous exercise, there are no elements x_1 and x_2 in the domain $\mathbb{R}$ of the function f such that $f(x_1) = f(x_2)$, which yields that f is injective. It is also clear that, for each element y of the range $\mathbb{R}$ of the function f, there exists an x in the domain $\mathbb{R}$ of f with the property $f(x) = y$, therefore, f is surjective. Thus, f is bijective.

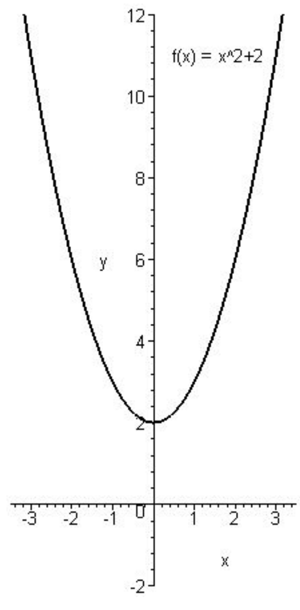
c) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x + 3$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = \frac{1}{2}x + 3$.

Similarly to the problems above, it is easy to verify that the function f is injective and surjective, therefore, it is bijective.

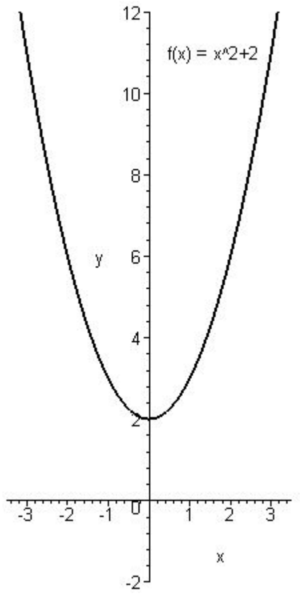
d) Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 2$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 2$.

Obviously, there are elements x_1 and x_2 in the domain $\mathbb{R}$ of the function f such that $f(x_1) = f(x_2)$. (For example, $f(-1) = f(1) = 3$.) Therefore f is not injective. Furthermore, there are elements y in the range $\mathbb{R}$ of the function f, for which there does not exist any x in the domain $\mathbb{R}$ of f with the property $f(x) = y$. (For example $y = 1$ has this property.) This means that f is not surjective. Since f is neither injective nor surjective, it is not bijective.

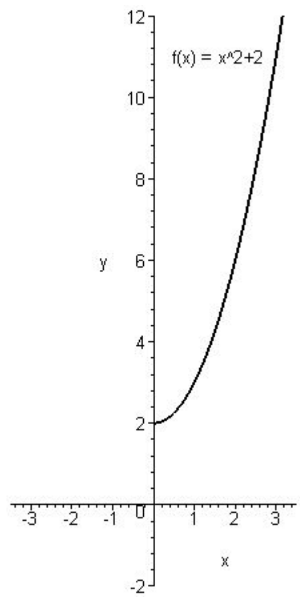
e) Let $f : \mathbb{R} \to [2, \infty[\,,\ f(x) = x^2 + 2$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to [2, \infty[\,,\ f(x) = x^2 + 2$.

It is easy to see that, for each element y of the range $[2, \infty[$ of f, there exists an x in the domain $\mathbb{R}$ of f such that $f(x) = y$, therefore, this function is surjective. However, similarly to the case above, there are elements x_1 and x_2 in the domain $\mathbb{R}$ of the function f such that $f(x_1) = f(x_2)$, which implies that f is not injective. Since f is not injective, it is not bijective.

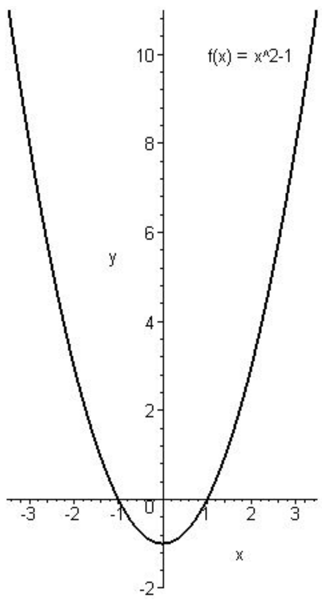
f) Let $f : [0, \infty[\to [2, \infty[\,,\ f(x) = x^2 + 2$. The graph of f has the form:



Graph of the function $f : [0, \infty[\to [2, \infty[\,,\ f(x) = x^2 + 2$.

Analogously to the exercise above, for each element y of the range $[0, \infty[$ of f, there exists an x in the domain $\mathbb{R}$ of f such that $f(x) = y$, therefore, f is surjective. Furthermore, there are no elements x_1 and x_2 in the domain $[0, \infty[$ of f such that $f(x_1) = f(x_2)$, which implies that f is injective. Since f is surjective and injective, it is also bijective.

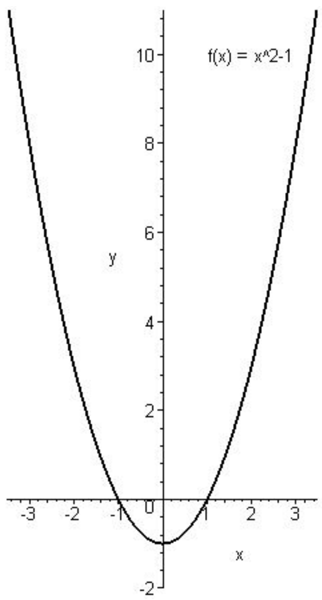
g) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

Similarly to the argumentation given in Exercise d above, it can be verified that the function f is neither injective nor surjective, therefore, it is not bijective.

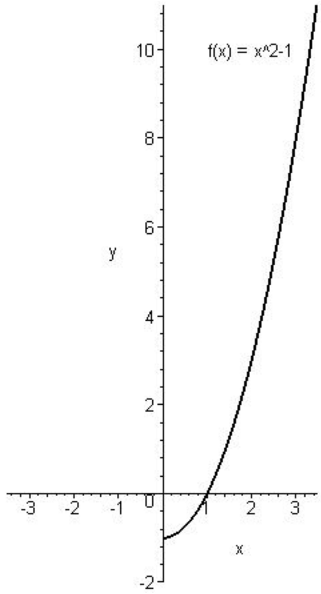
h) Let $f : \mathbb{R} \to [-1, \infty[$, $f(x) = x^2 - 1$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to [-1, \infty[$, $f(x) = x^2 - 1$.

Similarly to Exercise e, we obtain that the function f is surjective, but it is not injective, thus, it is not bijective.

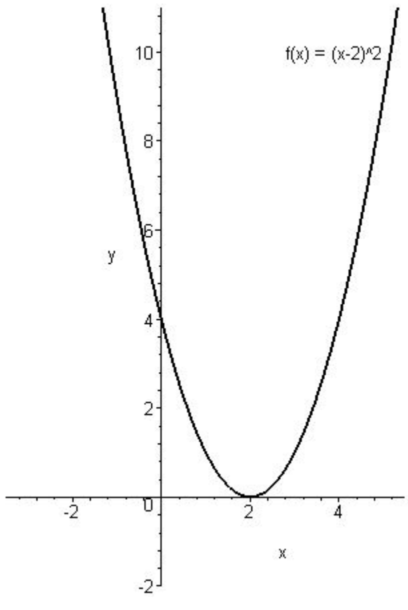
i) Let $f : [0, \infty[\to [-1, \infty[$, $f(x) = x^2 - 1$. The graph of f has the form:



Graph of the function $f : [0, \infty[\to [-1, \infty[\,,\ f(x) = x^2 - 1$.

Analogously to Exercise f, we may see that the function f is surjective and injective and this implies that it is also bijective.

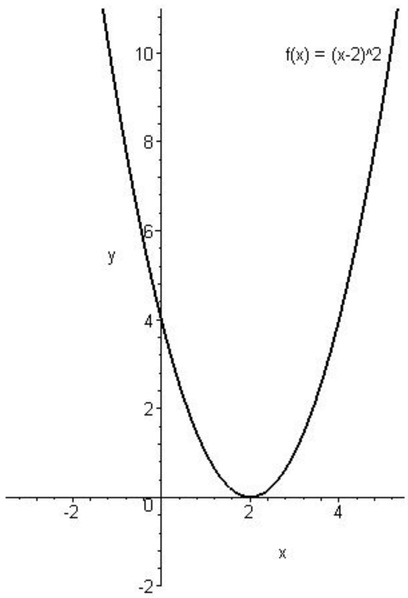
j) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (x-2)^2$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (x-2)^2$.

Similarly to the solution of the exercises above, we obtain that the function f is neither injective nor surjective, therefore, it is not bijective.

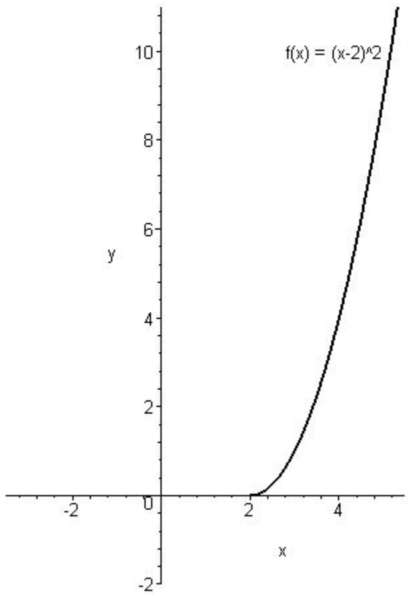
k) Let $f : \mathbb{R} \to [0, \infty[\,,\ f(x) = (x-2)^2$. The graph of f has the form:



Graph of the function $f : \mathbb{R} \to [0, \infty[\,,\ f(x) = (x-2)^2$.

It is easy to verify that the function f is surjective, but it is not injective, thus, it is not bijective.

l) Let $f : [2, \infty[\to [0, \infty[\,,\ f(x) = (x-2)^2$. The graph of f has the form:

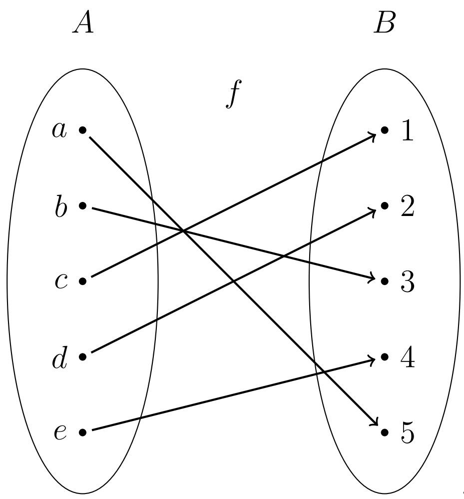


Graph of the function $f : [2, \infty[\to [0, \infty[\,,\ f(x) = (x-2)^2$.

It is easy to see that the function f is surjective and injective, therefore it is also bijective.

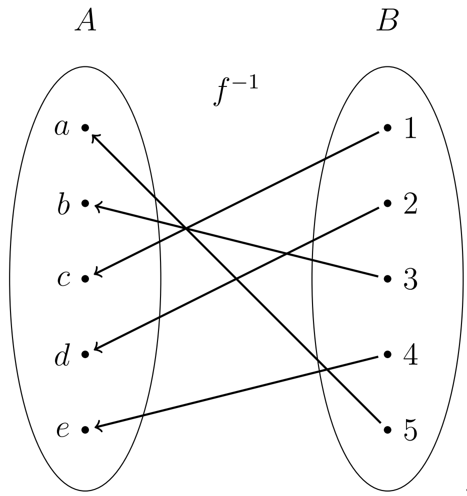
Solution of Exercise 1.9

a) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$ and let $f : A \to B$, $f(a) = 5$, $f(b) = 3$, $f(c) = 1$, $f(d) = 2$, $f(e) = 4$. We may illustrate f in the following form:



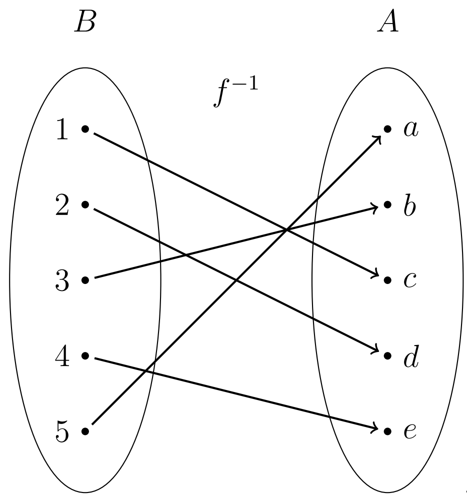
.

It is easy to see that f satisfies the requirements given in Definition 1.6, thus, it is a function. Furthermore, f is injective, thus, there exists its inverse. The inverse $f^{-1} : B \to A$ of f has the form:



,

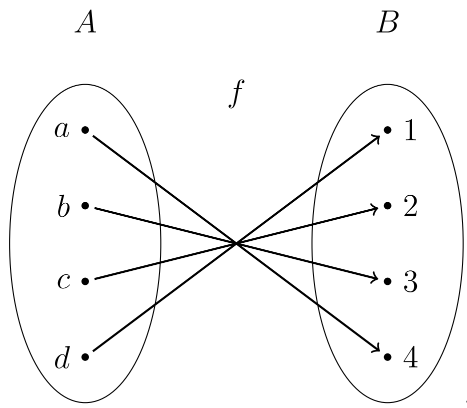
i. e.,



,

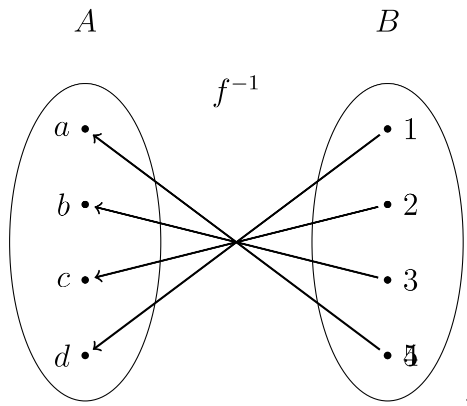
which means that $f^{-1} : B \to A$, $f^{-1}(1) = c$, $f^{-1}(2) = d$, $f^{-1}(3) = b$, $f^{-1}(4) = e$ and $f^{-1}(5) = a$.

b) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and let $f : A \to B$, $f(a) = 4$, $f(b) = 3$, $f(c) = 2$, $f(d) = 1$. We may illustrate f as:



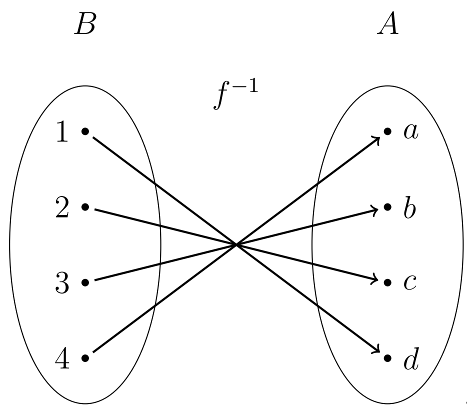
.

It is easy to see that f an injective function, thus, there exists its inverse. The inverse $f^{-1} : B \to A$ of f has the form:



,

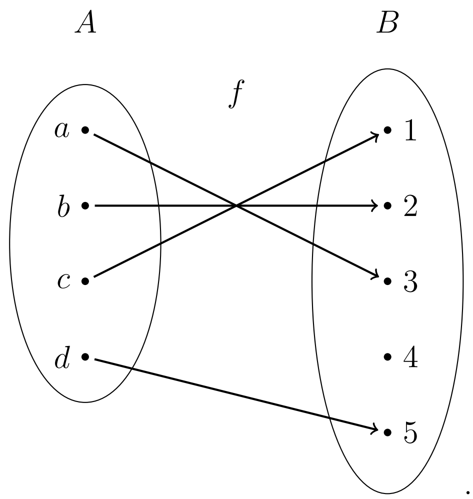
that is,



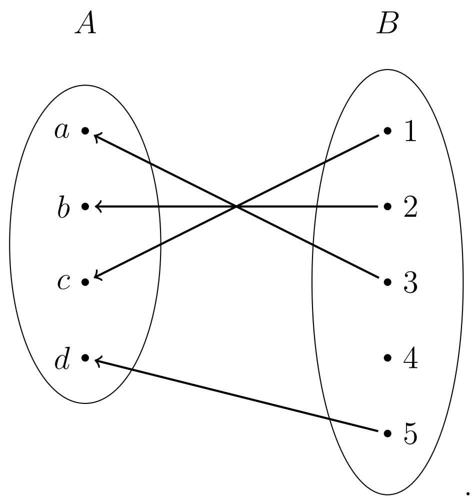
,

which means that $f^{-1} : B \to A$, $f^{-1}(1) = d$, $f^{-1}(2) = c$, $f^{-1}(3) = b$, $f^{-1}(4) = a$.

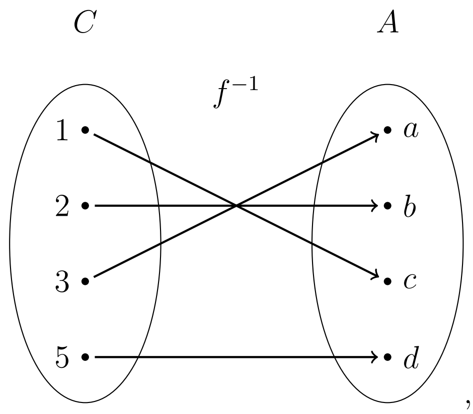
c) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4, 5\}$, $f : A \to B$, $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, $f(d) = 5$. A diagram of f is:



It is visible that f is a function and it is injective, therefore, it is invertible. Drawing 'mechanically' the correspondence between the elements of the sets B and A, we obtain:

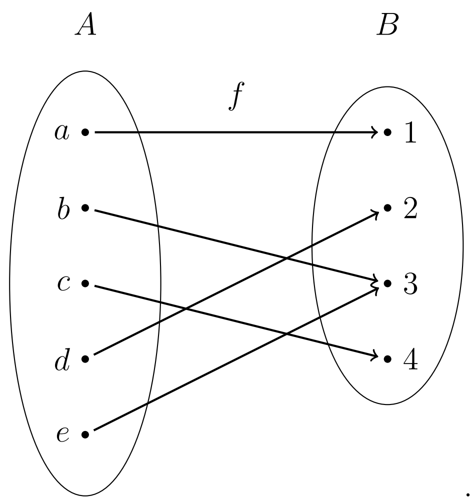


It is important to notify that the range of the function f is the set $C = \{1, 2, 3, 5\}$, therefore, this set will be the domain of the inverse function f^{-1} of f.



which gives that $f^{-1} : C \to A$ and $f^{-1}(1) = c$, $f^{-1}(2) = d$, $f^{-1}(3) = b$, $f^{-1}(4) = e$ and $f^{-1}(5) = a$.

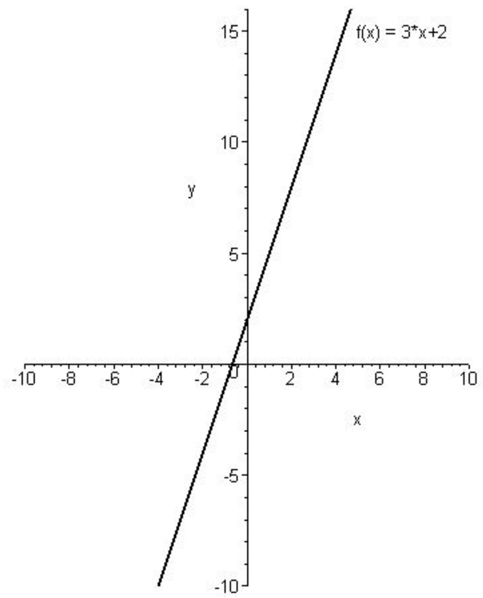
d) Let $A = \{a, b, c, d, e\}$, $B = \{1, 2, 3, 4\}$, $f : A \to B$, $f(a) = 1$, $f(b) = 3$, $f(c) = 4$, $f(d) = 2$, $f(d) = 3$. A diagram of f is:



It is easy to see that f is a function, but it is not injective, since $f(b) = f(d) = 3$. Therefore, the inverse of f does not exists.

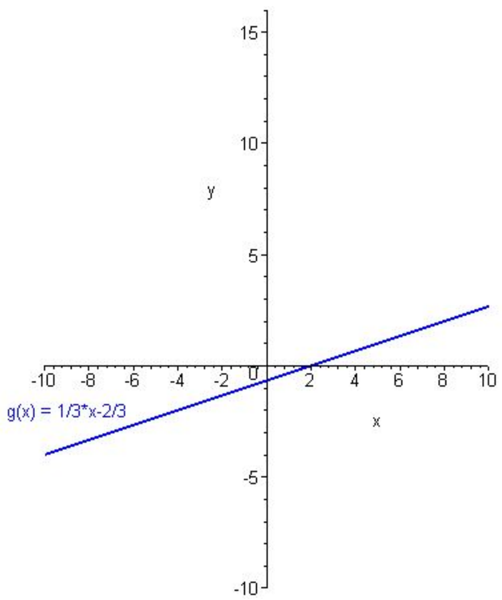
Solution of Exercise 1.10

a) The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$ is



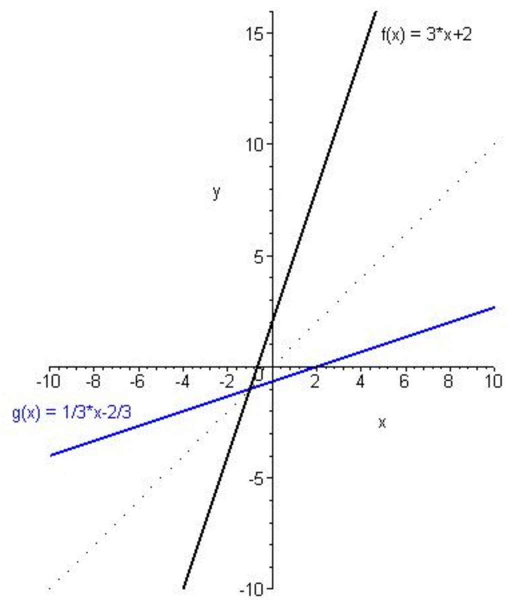
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$.

It is easy to see that this function is injective, therefore, it is invertible. Its inverse can be determined in the following form: by the definition of f, we have $y = 3x + 2$ for all $x \in \mathbb{R}$, which gives $x = \frac{1}{3}y - \frac{2}{3}$ for each $y \in \mathbb{R}$. Therefore, the inverse $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ of f is $g(y) = f^{-1}(y) = \frac{1}{3}y - \frac{2}{3}$ or, writing x instead of y, $g(x) = f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$. The graph of $g = f^{-1}$ is:



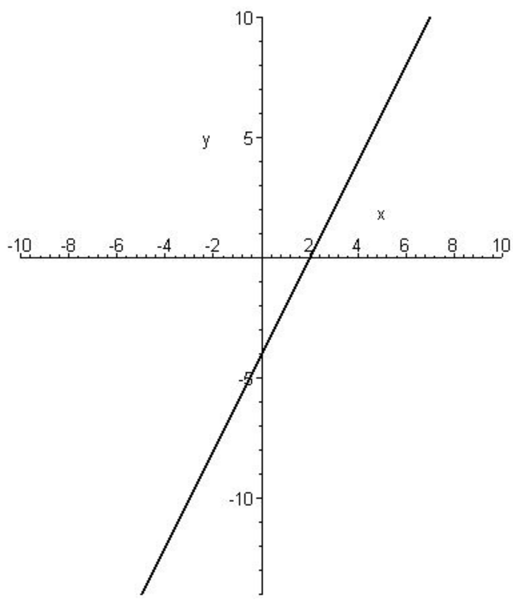
Graph of the function $g = f^{-1} : \mathbb{R} \to \mathbb{R}$, $g(x) = f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$.

The graph of f and its inverse in the same coordinate system



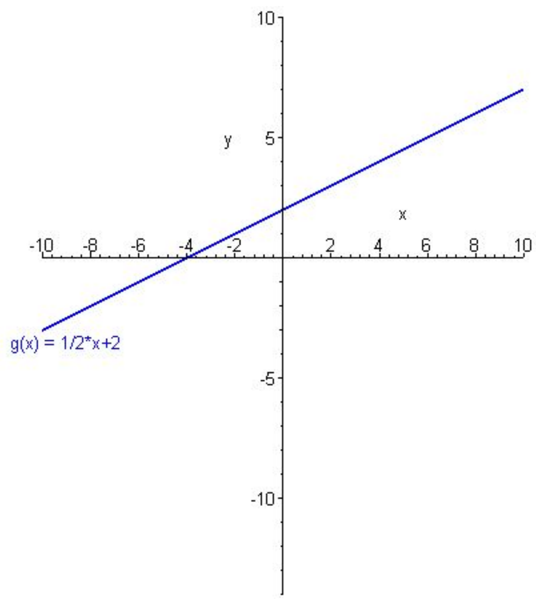
Graphs of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$ and its inverse $g = f^{-1} : \mathbb{R} \to \mathbb{R}$, $g(x) = f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$.

b) The graph of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 4$ is



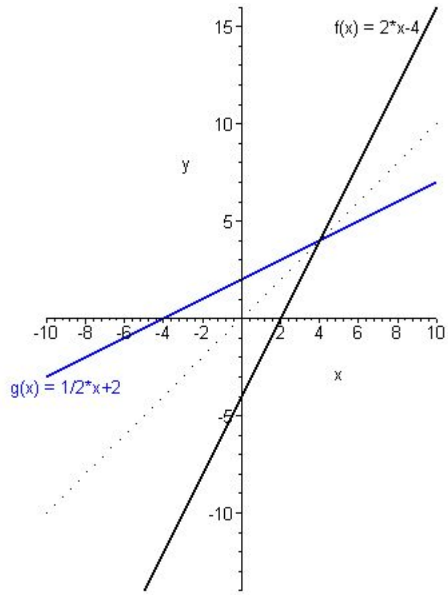
Graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 4$.

This function is injective, therefore, it is invertible. Its inverse can be obtained in the following form: $y = 2x - 4$ for all $x \in \mathbb{R}$, which yields $x = \frac{1}{2}y + 2$ for each $y \in \mathbb{R}$. The inverse $g = f^{-1} : \mathbb{R} \to \mathbb{R}$ of f reads as $g(y) = f^{-1}(y) = \frac{1}{2}y + 2$ or, writing x instead of y, $g(x) = f^{-1}(x) = \frac{1}{2}x + 2$. The graph of $g = f^{-1}$ is:



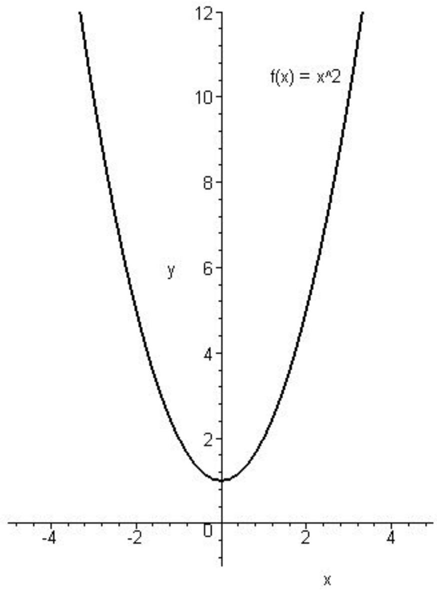
Graph of the function $g = f^{-1} : \mathbb{R} \to \mathbb{R}$, $g(x) = f^{-1}(x) = \frac{1}{2}x + 2$.

The graph of f and its inverse is:



Graphs of $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x-4$ and its inverse $g=f^{-1}:\mathbb{R}\to\mathbb{R}$, $g(x)=f^{-1}(x)=\frac{1}{2}x+2$.

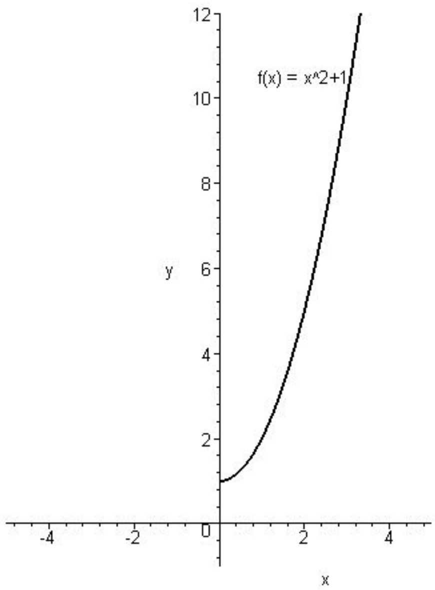
c) The graph of the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2+1$ is:



Graph of the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2+1$.

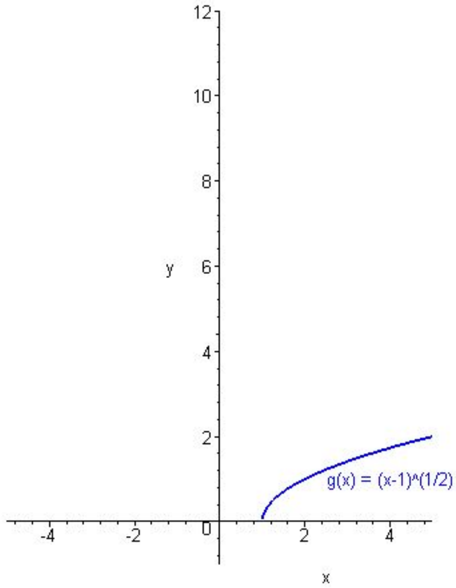
This function is not injective, therefore, it is not invertible.

d) The graph of $f : [0, \infty[\to \mathbb{R},\ f(x) = x^2 + 1$ is



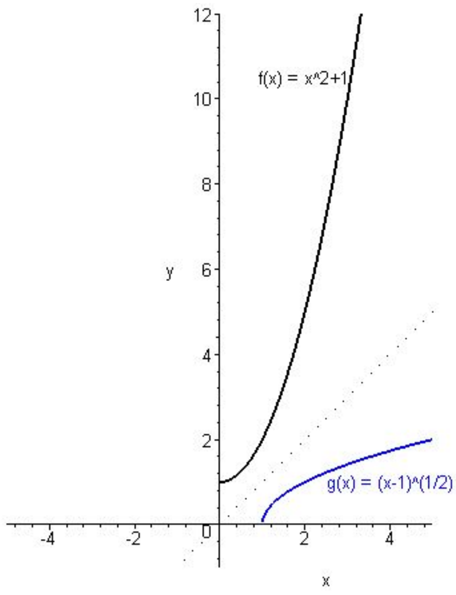
Graph of the function $f : [0, \infty[\to \mathbb{R},\ f(x) = x^2 + 1$.

This function is injective, thus, it is invertible. Its inverse can be determined by the following calculation. The equation $y = x^2 + 1$ for all $x \in \mathbb{R}$, gives $x = \sqrt{y-1}$ for each $y \in \mathbb{R}$. It is important to note that the range of f is the set $C = [1, \infty[$, therefore, this will be the domain of the inverse of f. The inverse $g = f^{-1} : [1, \infty[\to \mathbb{R}$ of f is $g(x) = f^{-1}(x) = \sqrt{y-1}$. Its graph is:



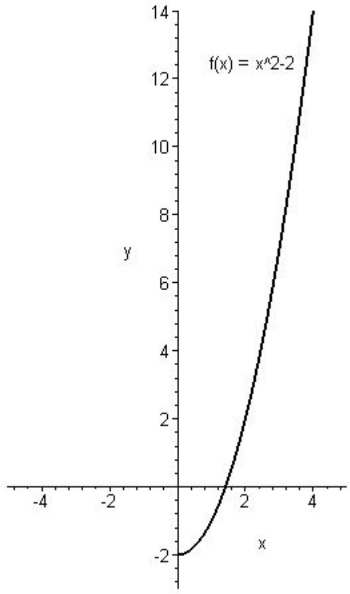
Graph of the function $g = f^{-1} : [1, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$.

The graphs of f and its inverse is:



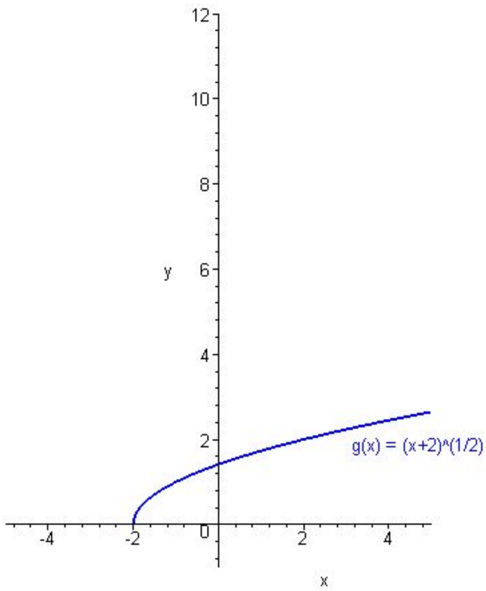
Graphs of $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2 + 1$ and its inverse $g = f^{-1} : [1, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = \sqrt{x-1} = (x-1)^{\frac{1}{2}}$.

e) The graph of $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2 - 2$ is:



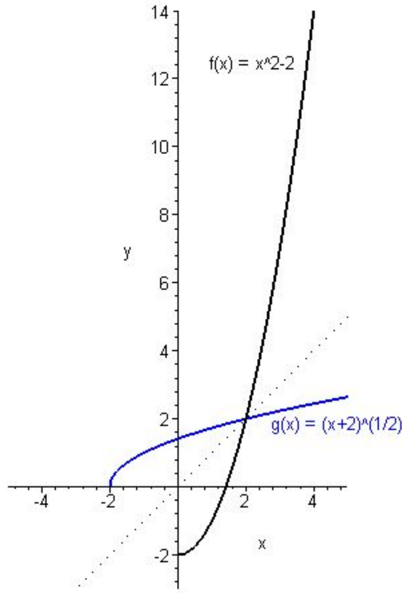
Graph of the function $f : [0, \infty[\to \mathbb{R},\ f(x) = x^2 - 2$.

The function f is injective, thus, it is invertible. The range of f is the set $C = [-2, \infty[$, thus, this set will be the domain of the inverse of f. The $g = f^{-1} : [-2, \infty[\to \mathbb{R}$ of f is $g(x) = f^{-1}(x) = \sqrt{x + 2}$. Its graph is:



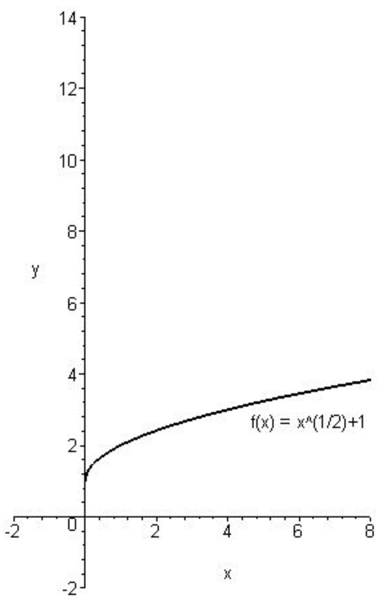
Graph of the function $g = f^{-1} : [-2, \infty[\to \mathbb{R},\ g(x) = f^{-1}(x) = \sqrt{x + 2} = (x + 2)^{\frac{1}{2}}$.

The graphs of f and g are:



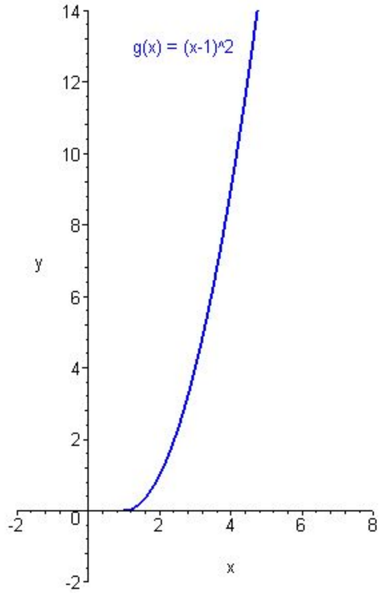
Graphs of $f : [0, \infty[\to \mathbb{R}$, $f(x) = x^2 - 2$ and its inverse $g = f^{-1} : [-2, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$.

f) The graph of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} + 1$ is:



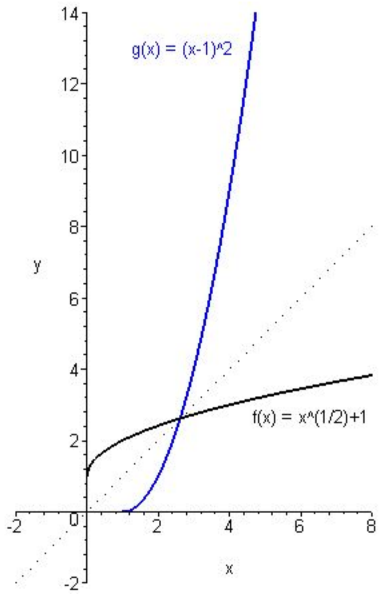
Graph of the function $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} + 1$.

The function f is injective, thus, it is invertible. The range of f is the set $C = [1, \infty[$, thus, this set will be the domain of the inverse of f. The $g = f^{-1} : [1, \infty[\to \mathbb{R}$ of f is $g(x) = f^{-1}(x) = (x-1)^2$. Its graph is:



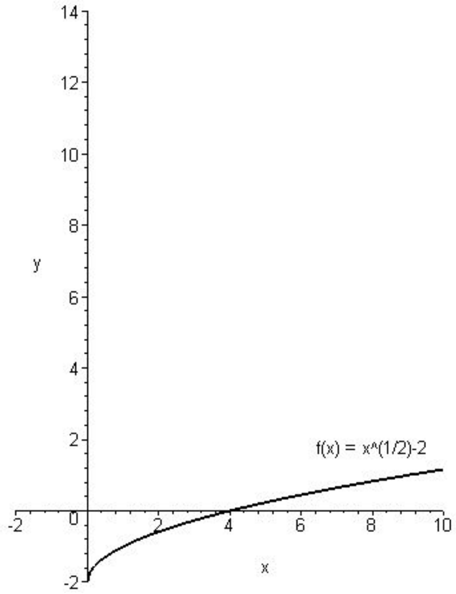
Graph of the function $g = f^{-1} : [1, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = (x-1)^2$.

The graphs of f and g are:



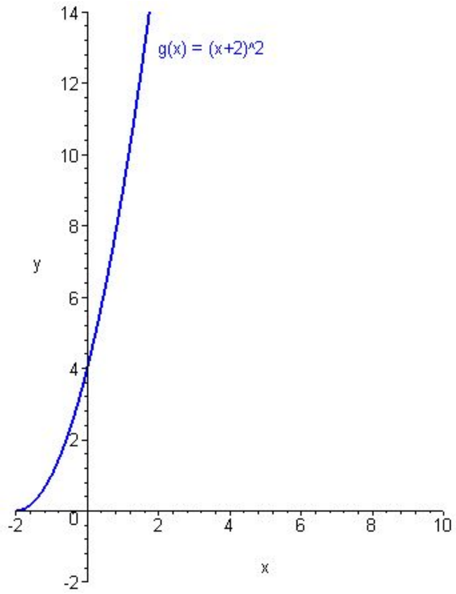
Graphs of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} + 1$ and its inverse $g = f^{-1} : [1, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = (x-1)^2$.

g) The graph of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} - 2$ is:



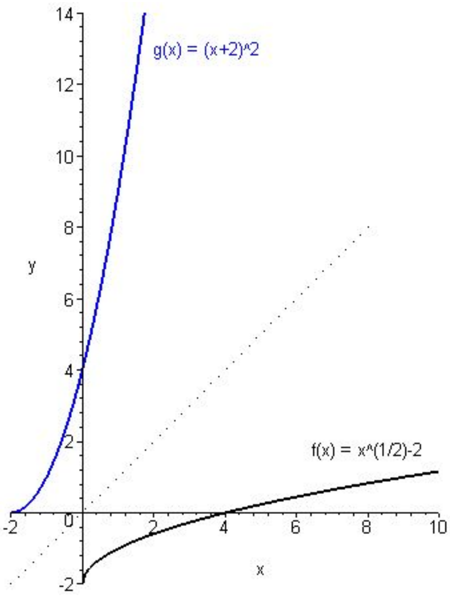
Graph of the function $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} - 2$.

The function f is injective, thus, it is invertible. The range of f is the set $C = [-2, \infty[$, thus, this set will be the domain of the inverse of f. The $g = f^{-1} : [-2, \infty[\to \mathbb{R}$ of f is $g(x) = f^{-1}(x) = (x + 2)^2$. Its graph is:



Graph of the function $g = f^{-1} : [-2, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = (x+2)^2$.

The graphs of f and g are:



Graphs of $f : [0, \infty[\to \mathbb{R}$, $f(x) = \sqrt{x} - 2$ and its inverse $g = f^{-1} : [-2, \infty[\to \mathbb{R}$, $g(x) = f^{-1}(x) = (x+2)^2$.

5.2 Results of the Exercises of Chapter 2

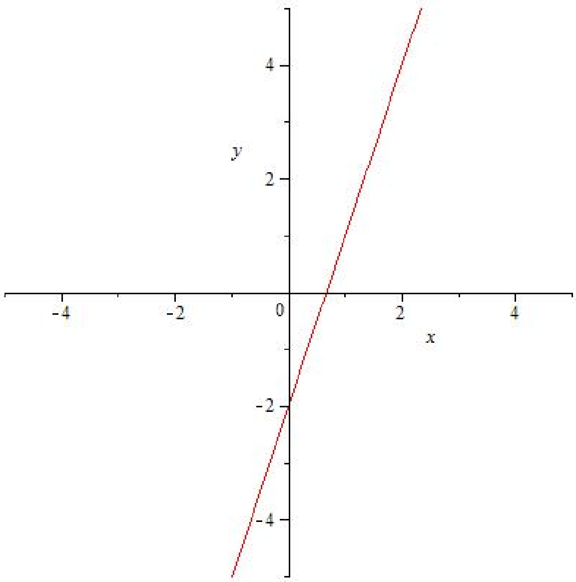
Solution of Exercise 2.1:

The formula defining the corresponding functions is:

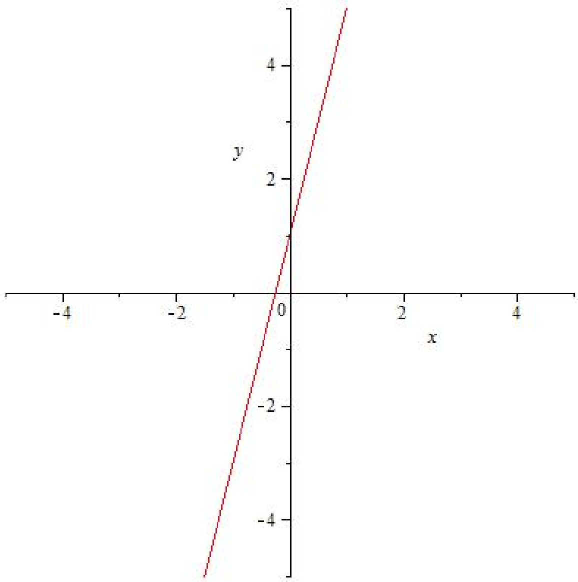
a) $f(x) = 3x - 2$ b) $f(x) = 4x + 1$
c) $f(x) = -2x + 5$ d) $f(x) = -5x - 4$

The graph of the corresponding function is:

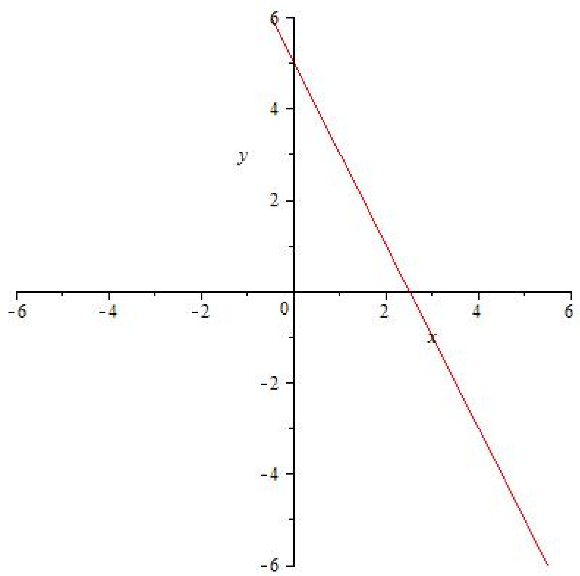
a)



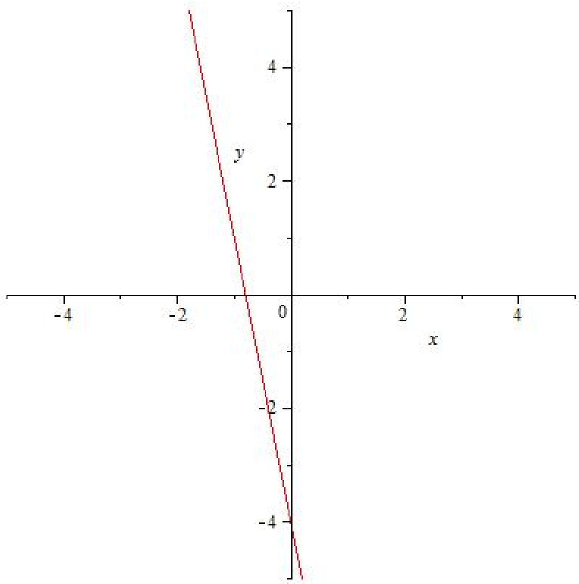
b)



c)



d)



Solution of Exercise 2.2:

a) $f(x) = 2x - 1$	b) $f(x) = 3x + 2$
c) $f(x) = -x + 6$	d) $f(x) = 3x + 7$
e) $f(x) = 4$	f) $f(x) = -2x + 9$
g) $f(x) = -7x + 2$	h) $f(x) = 5x - 4$
i) $f(x) = -3x + 7$	j) $f(x) = 4x + 2$
k) $f(x) = -3x + 11$	l) $f(x) = 4x + 7$
m) $f(x) = \sqrt{2}x + \sqrt{3}$	n) $f(x) = 2x - \sqrt{5}$

Solution of Exercise 2.3

a) $f(x) = 2x + 3$
b) $f(x) = -x + 5$
c) $f(x) = -3x + 5$
d) $f(x) = 5x + 1$
e) $f(x) = -7x + 3$
f) $f(x) = -2x + 7$
g) $f(x) = -5x + 11$
h) $f(x) = -8x + 4$
i) $f(x) = 11x - 7$
j) $f(x) = 5x - 21$
k) $f(x) = 3x + 7$
l) $f(x) = -x + 1$
m) $f(x) = x + 2$
n) $f(x) = 4$
o) $f(x) = 3x$
p) $f(x) = -7x$
q) $f(x) = \frac{1}{2}x - 3$
r) $f(x) = 3x + 4$
s) $f(x) = \sqrt{2}x + \sqrt{3}$
t) $f(x) = 3x - \sqrt{2}$

Solution of Exercise 2.4:

1). The canonical form of the corresponding functions is:

a) $f(x) = x^2 - 4$
b) $f(x) = \left(x - \frac{5}{2}\right)^2 + \frac{1}{4}$
c) $f(x) = -(x-3)^2 + 9$
d) $f(x) = -(x-2)^2 + 1$
e) $f(x) = (x - (-2))^2 + 4$
f) $f(x) = -x^2 - 4$
g) $f(x) = 2x^2 - 2$
h) $f(x) = 3(x-1)^2 - 3$
i) $f(x) = -2x^2 - 4$
j) $f(x) = -3(x-1)^2 - 3$
k) $f(x) = -2(x - (-1))^2$
l) $f(x) = 3(x-2)^2$
m) $f(x) = (x-3)^2 - 4$
n) $f(x) = -4x^2 + 1$
o) $f(x) = (x-5)^2$
p) $f(x) = -x^2 + 16$

2). The factorized form of the corresponding function (provided that $\Delta > 0$) is:

a) $f(x) = (x-2)(x+2)$	b) $f(x) = (x-2)(x-3)$
c) $f(x) = -x(x-6)$	d) $f(x) = -(x-1)(x-3)$
e) $-$	f) $-$
g) $f(x) = 2(x-1)(x+1)$	h) $f(x) = 3x(x-2)$
i) $-$	j) $-$
k) $f(x) = -2(x+1)^2$	l) $f(x) = 3(x-2)^2$
m) $f(x) = (x-1)(x-5)$	n) $f(x) = -4\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$
o) $f(x) = (x-5)^2$	p) $f(x) = -(x-4)(x+4)$

3). The Y-intercept is denoted by C, and the X-intercepts (if they exist at all) by A and B

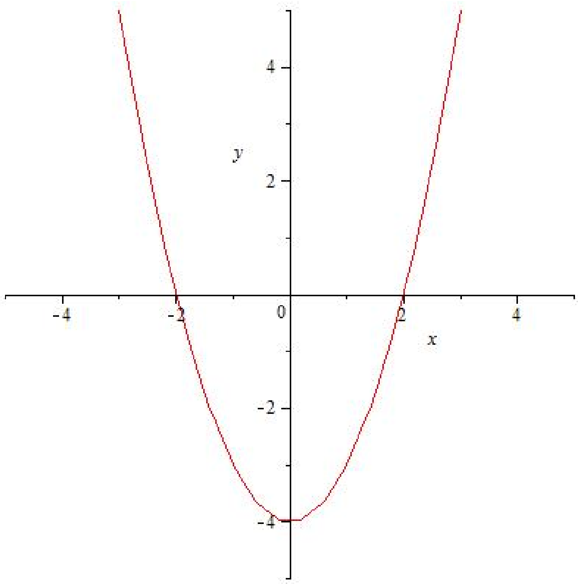
a) $C(0,-4),\ A(-2,0),\ B(2,0)$	b) $C(0,6),\ A(2,0),\ B(3,0)$
c) $C(0,0),\ A(0,0),\ B(6,0)$	d) $C(0,-3),\ A(1,0),\ B(3,0)$
e) $C(0,8),$ no x-intercepts	f) $C(0,-4),$ no x-intercepts
g) $C(0,-2),\ A(-1,0),\ B(1,0)$	h) $C(0,0),\ A(0,0),\ B(2,0)$
i) $C(0,-4),$ no x-intercepts	j) $C(0,-6),$ no x-intercepts
k) $C(0,-2),\ A(-1,0)$	l) $C(0,12),\ A(2,0)$
m) $C(0,5),\ A(1,0),\ B(5,0)$	n) $C(0,1),\ A\left(-\frac{1}{2},0\right),\ B\left(\frac{1}{2},0\right)$
o) $C(0,25),\ A(5,0)$	p) $C(0,16),\ A(-4,0),\ B(4,0)$

4). The vertex of the graph of the corresponding function is

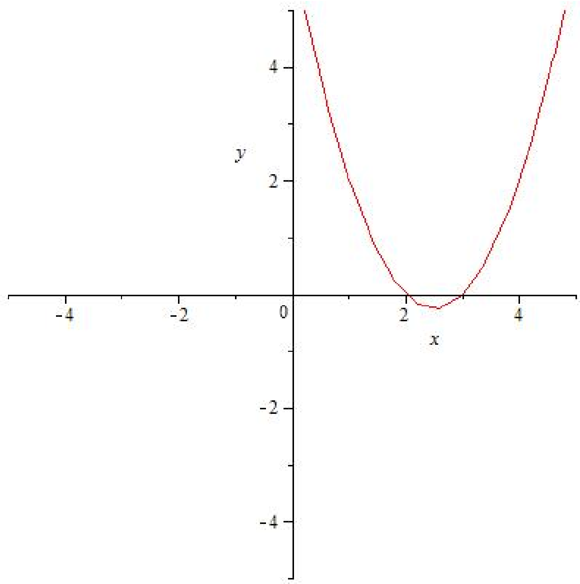
a) $V(0,-4)$	b) $V\left(\frac{5}{2},-\frac{1}{4}\right)$
c) $V(3,9)$	d) $V(2,1)$
e) $V(-2,4)$	f) $V(0,-4)$
g) $V(0,-1)$	h) $V(1,-3)$
i) $V(0,-4)$	j) $V(1,-3)$
k) $V(-1,0)$	l) $V(2,0)$
m) $V(3,-4)$	n) $V(0,1)$
o) $V(5,0)$	p) $V(0,16)$

5). The graph of f is

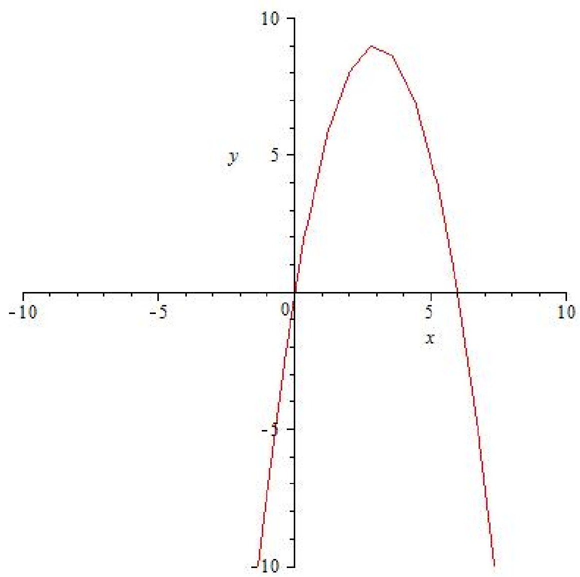
a)



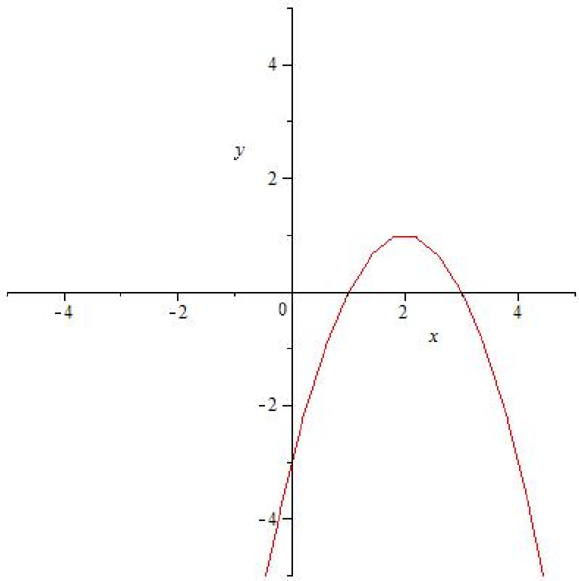
b)



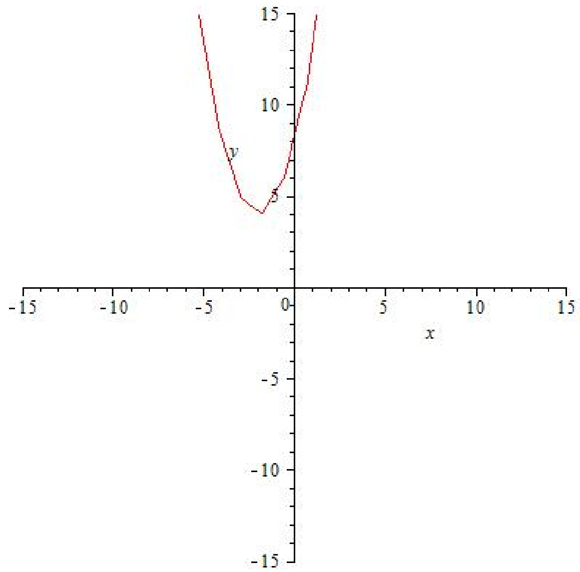
c)


10
5
y
-10
-5
0
5
10
x
-5
-10

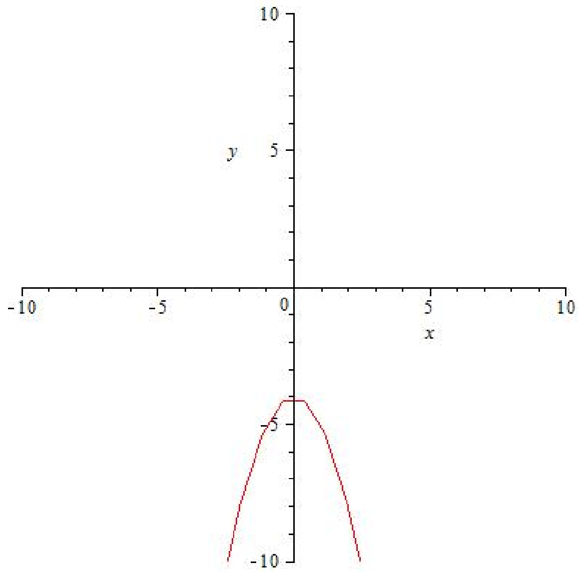
d)


4
2
y
-4
-2
0
2
4
x
-2
-4

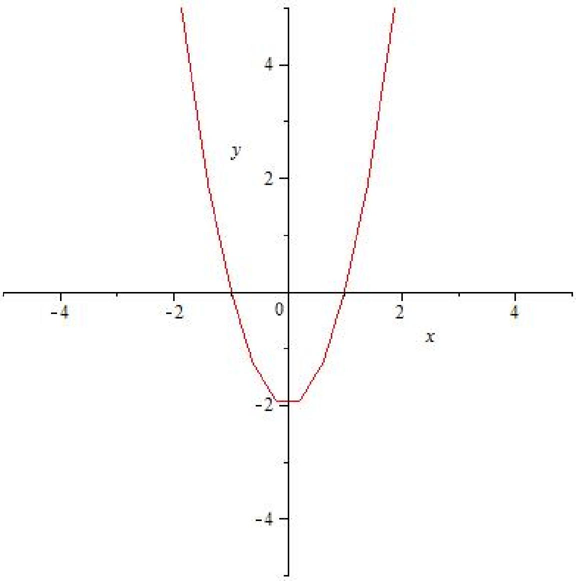
e)


15
10
y
5
-15
-10
-5
0
5
10
15
x
-5
-10
-15

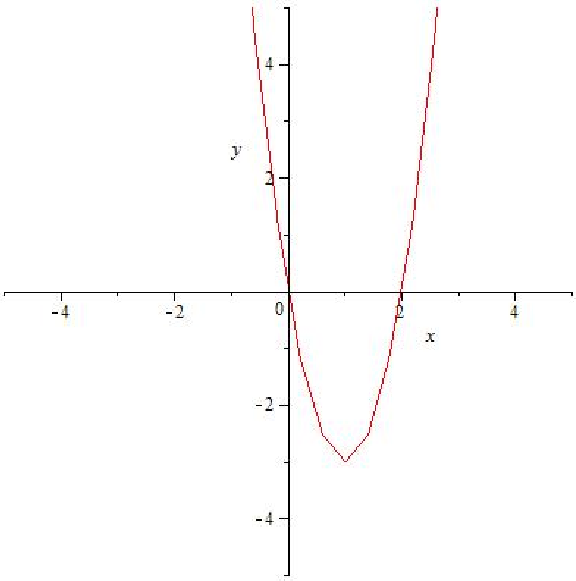
f)


10
y
5
-10
-5
0
5
10
x
-5
-10

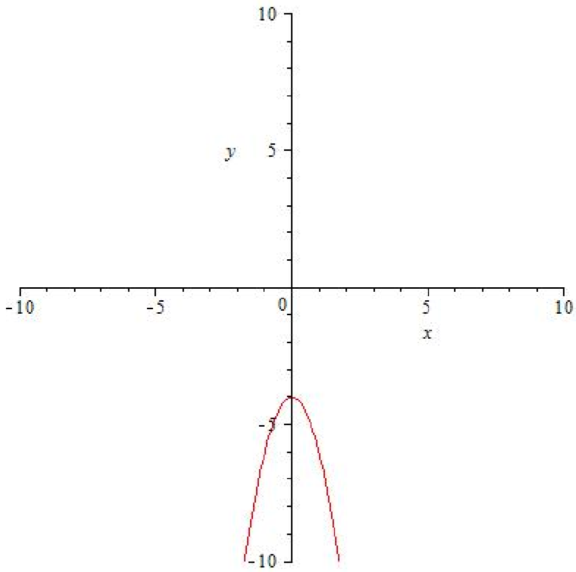
g)


y
x
4
2
-2
-4
-4
-2
0
2
4

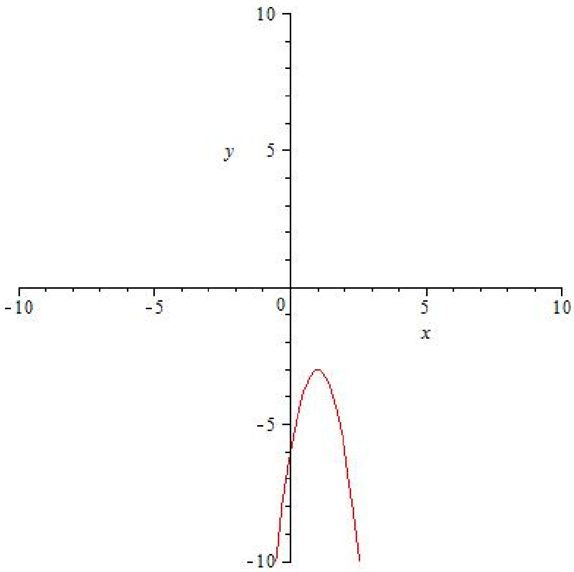
h)


y
x
4
2
-2
-4
-4
-2
0
2
4

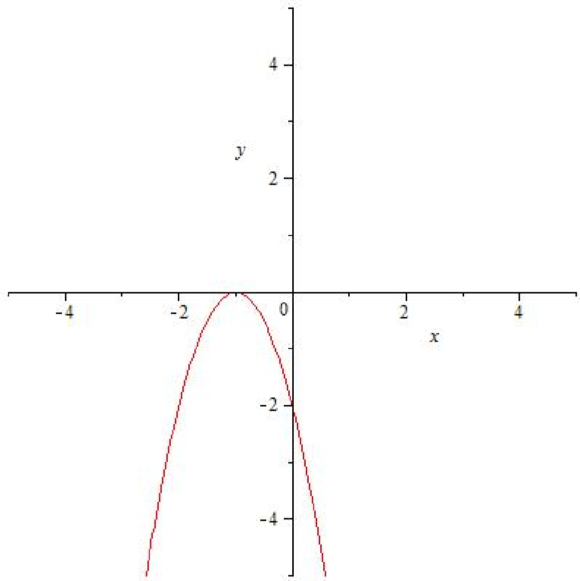
i)


y
x
10
5
-10
-10
-5
0
5
10

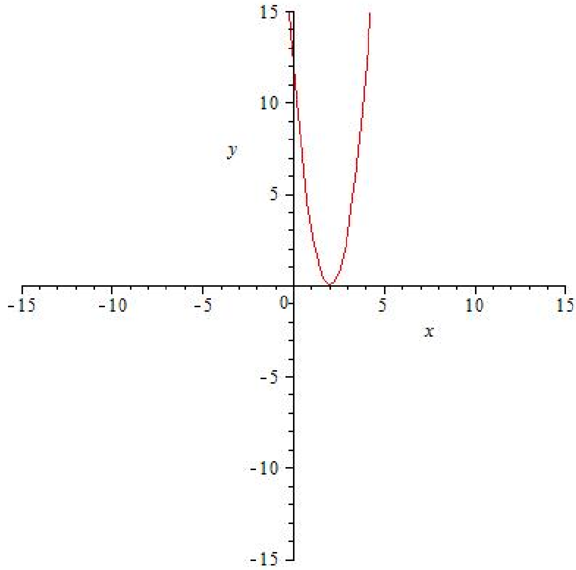
j)


y
x
10
5
-5
-10
-10
-5
0
5
10

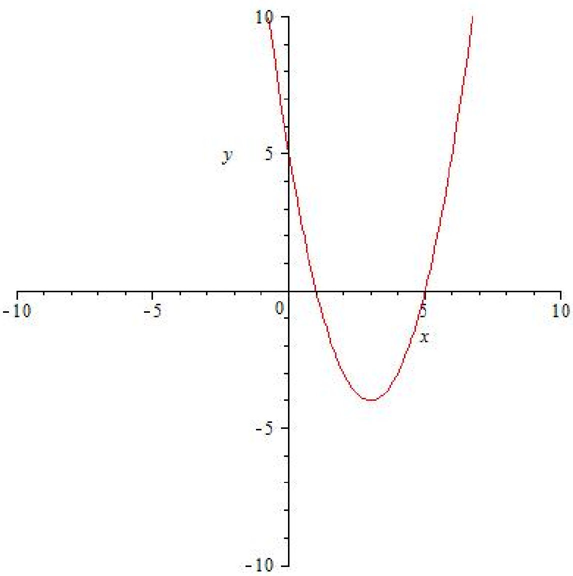
k)


y
4
2
-4
-2
0
2
4
x
-2
-4

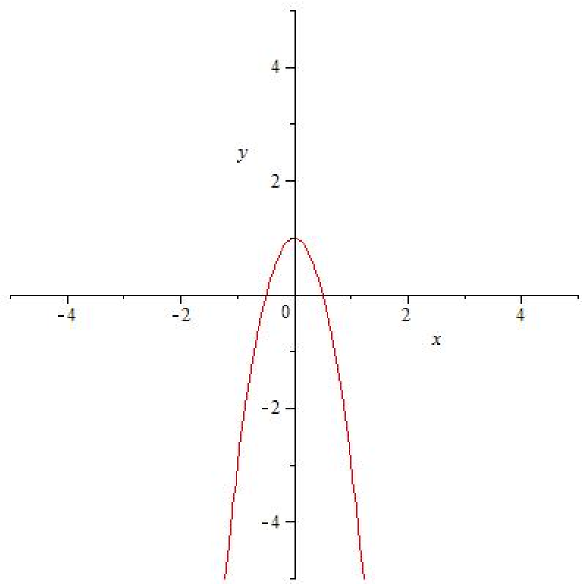
l)


15
10
y
5
-15
-10
-5
0
5
10
15
x
-5
-10
-15

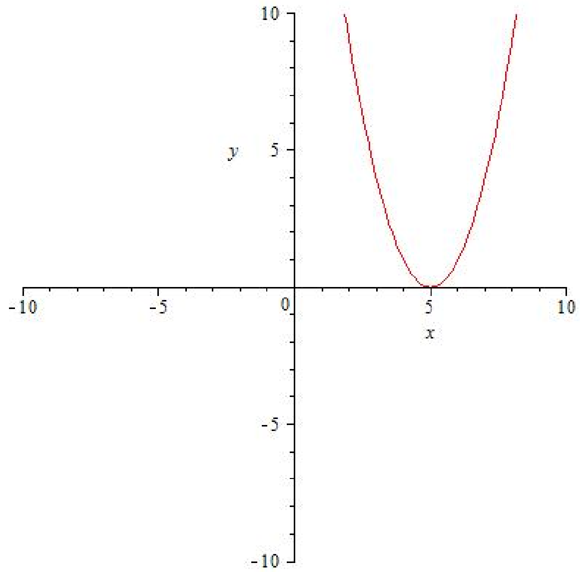
m)


10
y
5
-10
-5
0
5
10
x
-5
-10

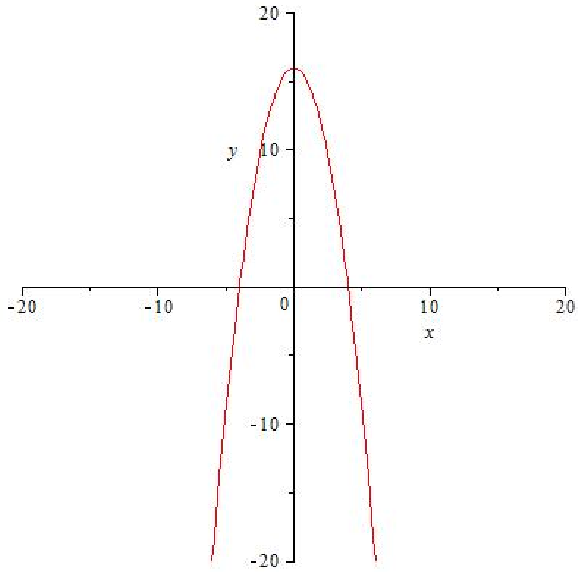
n)


4
y
2
-4
-2
0
2
4
x
-2
-4

o)



p)



Solution of Exercise 2.5: The quotient $q(x)$ and the remainder $r(x)$ of the corresponding division is

a) $q(x) = x + 3,$	$r(x) := -2x - 1$
b) $q(x) = x^3 - 2x^2 + x + 5,$	$r(x) = -3$
c) $q(x) = x^3 - 2x - 4,$	$r(x) = -12x + 10$
d) $q(x) = x^3 - x - 1,$	$r(x) = -5x + 3$
e) $q(x) = x^3 + 2x + 1,$	$r(x) = 0$
f) $q(x) = x^4 + 2x^3 - 8x - 16,$	$r(x) = 0$
g) $q(x) = x^3 + x^2 - x - 7,$	$r(x) = -20x + 30$
h) $q(x) = x^3 + 7x^2 + 28x + 126,$	$r(x) = 574x - 250$
i $q(x) = x^3 + 2x^2 - 4,$	$r(x) = -19x + 14$
j) $q(x) = x + 1,$	$r(x) = -3x^3 + 5x^2 - 2x - 1$
k) $q(x) = x^5 + 4x^4 + 2x^3 - 3x^2 - x - 4,$	$r(x) = 6$
l) $q(x) = x^5 + 2x^4 - 4x^3 - x^2 + 3x - 6,$	$r(x) = 16$
m) $q(x) = x^5 + 5x^4 + 8x^3 + 11x^2 + 24x + 45,$	$r(x) = 100$
n) $q(x) = x^5 + x^4 - 4x^3 + 3x^2 - 4x + 5,$	$r(x) = 0$
o) $q(x) = x^5 - 2 * x^3 + x^2 - x,$	$r(x) = 10$
p) $q(x) = x^4 + 3x^3 - x^2 - 2x + 1,$	$r(x) = -5x + 11$
q) $q(x) = x^5 + x^4 - 3x^3 - 4x^2 - 4x - 1,$	$r(x) = -3$
r) $q(x) = x^5 - x^4 - 3x^3 + 2x^2 - 2x + 5,$	$r(x) = -7$
s) $q(x) = x^5 + 2x^4 - x^2 - 2x - 1,$	$r(x) = -4$
t) $q(x) = x^5 - 2x^4 - x^2 + 2x - 1,$	$r(x) = 0$
u) $q(x) = x^5 + 3x^4 + 5x^3 + 14x^2 + 42x + 129,$	$r(x) = 385$
v) $q(x) = x^5 - 3x^4 + 5x^3 - 16x^2 + 48x - 141,$	$r(x) = 421$
w) $q(x) = x^4 + x^3 - 2x^2 - 2x - 4,$	$r(x) = -3x - 6$
x) $q(x) = x^4 - 3x^2 - x - 3,$	$r(x) = 2x - 5$
y) $q(x) = x^4 - x^3 - 2x^2 - 2,$	$r(x) = 5x - 4$
z) $q(x) = x^4 - 2x^3 + x^2 - 5x + 11,$	$r(x) = -24x + 9$

Solution of Exercise 2.6

a) 143	b) 0	c) 729
d) -6012	e) -3645	f) -256
g) 48	h) 36	i) 64
j) -80	k) 70	l) -110
m) 0	n) 0	o) 1120
p) -24	q) -60	r) 0
s) 0	t) 0	u) -120
v) -16	w) 0	x) 192
y) 0	z) -28	ω) -240

5.3 Results of the Exercises of Chapter 3

Solution of Exercise 3.1

a) $-\frac{1}{2}$	b) $\frac{\sqrt{3}}{2}$	c) 1
d) $-\frac{\sqrt{3}}{2}$	e) $-\frac{\sqrt{2}}{2}$	f) $\sqrt{3}$
g) $-\frac{\sqrt{3}}{2}$	h) $\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}$	i) $-\frac{\sqrt{2}}{2}$
j) $-\frac{1}{2}$	k) $-\frac{\sqrt{3}}{2}$	l) $-\frac{1}{2}$
m) $-\sqrt{3}$	n) $-\frac{\sqrt{3}}{3}$	o) $\frac{\sqrt{2}}{2}$
p) $\frac{\sqrt{3}}{3}$	q) $\frac{\sqrt{2}}{2}$	r) $-\frac{\sqrt{2}}{2}$
s) $-\frac{\sqrt{3}}{2}$	t) $\frac{\sqrt{3}}{3}$	u) $-\frac{1}{2}$
v) 0	w) $-\frac{\sqrt{2}}{2}$	x) 0
y) not defined	z) 0	ω) not defined

Solution of Exercise 3.2:

a) $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$	b) $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$	c) $2+\sqrt{3}$
d) $2-\sqrt{3}$	e) $\frac{1}{2}\sqrt{2-\sqrt{2}}$	f) $\frac{1}{2}\sqrt{2+\sqrt{2}}$
g) $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$	h) $\frac{-\sqrt{2}(\sqrt{3}-1)}{4}$	i) $-2-\sqrt{3}$
j) $-2+\sqrt{3}$	k) $\frac{1}{2}\sqrt{2+\sqrt{2}}$	l) $\frac{1}{2}\sqrt{2-\sqrt{2}}$

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